UNIVERSIDADE ESTADUAL DE CAMPINAS

Instituto de Matemática, Estatística e Computação Científica

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Higgs-Hermite-Einstein metrics and Spin(7)−instantons over ACyl manifolds

Métricas Higgs-Hermite-Einstein e Spin(7)−instantons sobre variedades ACil

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

Supervisor: Henrique Nogueira de Sá Earp

Este exemplar corresponde à versão final da Tese defendida pelo aluno Pedro Manfrim Magalhães de Paula e orientada pelo Prof. Dr. Henrique Nogueira de Sá Earp.

Campinas

2019

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

P281h Paula, Pedro Manfrim Magalhães de, 1991-
Higgs-Hermite-Einstein metrics and Spin(7)-instantons over ACyl manifolds / Pedro Manfrim Magalhães de Paula. – Campinas, SP : [s.n.], 2019.

Orientador: Henrique Nogueira de Sá Earp.
Tese (doutorado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Conexões (Matemática). 2. Geometria complexa. 3. Higgs, Fibrados de. I. Sá Earp, Henrique Nogueira de, 1981-. II. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. III. Título.

Informações para Biblioteca Digital

Título em outro idioma: Métricas Higgs-Hermite-Einstein e Spin(7)-instantons sobre variedades ACil
Palavras-chave em inglês:
Connections (Mathematics)
Complex geometry
Higgs bundles
Área de concentração: Matemática
Titulação: Doutor em Matemática
Banca examinadora:
Henrique Nogueira de Sá Earp [Orientador]
Marcos Benevenuto Jardim
Gonçalo Marques Fernandes de Oliveira
Andrew James Clarke
Matheus Brioschi Herkenhoff Vieira
Data de defesa: 28-06-2019
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
- ORCID do autor: https://orcid.org/0000-0002-8646-6738
- Currículo Lattes do autor: http://lattes.cnpq.br/8450502663168488

Tese de Doutorado defendida em 28 de junho de 2019 e aprovada

pela banca examinadora composta pelos Profs. Drs.

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A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

Acknowledgements

First I would like to express my deepest appreciation to Professor Henrique N. Sá Earp for the patience and encouragement throughout the doctorate. It is his assistance that made all of this work possible.

I would also like to extend my deepest gratitude to the rest of my thesis committee: Marcos Jardim, Gonçalo Oliveira, Andrew Clarke and Matheus Vieira; whose advices helped me improve a lot this final work.

I am grateful to the Universidade Estadual de Campinas, particularly the Instituto de Matemática, Estatística e Computação Científica, for the opportunity provided since my arrival as an undergraduate. I very much appreciate all the community, including all professors and employees that I had the pleasure of knowing during all these years, for being part of my academic and cultural development.

I also wish to thank my friends and colleagues at UNICAMP for the joyful moments we shared together. Especially, I would like to thank Daniel, Eduardo and Felipe for the stimulating conversations we had all these years.

Finally, I must not forget to thank my family for their unconditional support that helped me become what I am today, especially my parents Agostinho and Maria Cecília that spared no effort so that I reached this stage of my life, and my brother Daniel whose unending dialogues promoted a lot of my personal growth.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001 and by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) - Process 155046/2016-0

Resumo

Desde os trabalhos iniciais de Donaldson sobre instantons Yang-Mills era aparente a relação entre instantons Hermite-Yang-Mills e métricas Hermite-Einstein. Recentemente, motivados por uma relação semelhante entre G_2 −instantons e métricas Hermite-Einstein, Sá Earp e Walpuski introduziram um método para construir novos exemplos de G_2 −instantons sobre somas conexas torcidas. Baseado nisso, o objetivo desta tese é propor uma construção análoga para Spin(7) −instantons usando métricas Higgs-Hermite-Einstein. Primeiramente, mostramos como reduzir a equação de Spin(7) −instantons a aquela de métricas Higgs-Hermite-Einstein. Para isso seguimos a idéa de redução dimensional introduzida por Hitchin. Então provamos a existência de métricas Higgs-Hermite-Einstein para fibrados de Higgs assintoticamente translacionalmente invariantes sobre variedades Kähler assintoticamente cilindricas. Isso é alcançado adaptando o método de continuidade de Uhlenbeck-Yau para este contexto não-compacto usando os espaços de Hölder com peso exponencial adequados. Com isso concluimos o primeiro passo na contrução proposta de Spin(7) −instantons sobre somas conexas torcidas, que se espera produzir novos exemplos de Spin(7) −instantons.

Palavras-chave: Conexões(Matemática), Geometria complexa, Fibrados de Higgs.

Abstract

Since Donaldson's early works on Yang-Mills instantons it was apparent the relation between Hermite-Yang-Mills instantons and Hermite-Einstein metrics. Recently, motivated by a similar relation between G_2-instantons and Hermite-Einstein metrics, Sá Earp and Walpuski introduced a method to construct new examples of G_2-instantons over twisted connected sums. Based in this, the objective of this thesis is to propose an analogous construction for $\mathrm{Spin}(7)-$instantons using Higgs-Hermite-Einstein metrics. Firstly, we show how to reduce the $\mathrm{Spin}(7)-$instanton equation to that of Higgs-Hermitian-Einstein metrics. For this we follow the idea of dimensional reduction introduced by Hitchin. Then we prove the existence of Higgs-Hermite-Einstein metrics for asymptotically translation invariant Higgs bundles over asymptotically cylindrical Kähler manifolds. This is achieved by adapting Uhlenbeck-Yau's continuity method to this non-compact setting using the suitable Hölder spaces with exponential weights. With this we conclude the first step in the proposed construction of $\mathrm{Spin}(7)-$instantons over twisted connected sum, which is expected to produce new examples of $\mathrm{Spin}(7)-$instantons.

Keywords: Connections(Mathematics), Complex geometry, Higgs bundles.

Contents

Introduction

In this work we study Higgs-Hermite-Einstein metrics over ACyl Kähler manifolds and their relation to $\mathrm{Spin}(7)-$instantons. Our main motivation is the construction of $\mathrm{Spin}(7)-$instantons in a manner similar to what was done in (EARP; WALPUSKI, 2015) for G_2-instantons.

If $(\mathcal{E}, \theta)$ is a Higgs bundle over a Kähler manifold (X, ω) and H is a Hermitian metric on $\mathcal{E}$, the Hitchin-Simpson connection is defined as

$$D = \nabla_H + \theta + \theta^*$$

where ∇_H is the Chern connection associated to H and θ^* is the adjoint of θ with respect to H. Denoting by F_D the curvature of D and Λ the contraction by ω, we say that H satisfies the (weak) Higgs-Hermite-Einstein equation if

$$\Lambda F_D = \lambda \cdot \mathrm{Id}$$

for a constant (function) $\lambda \in \mathbb{R}$.

These metrics have a fundamental role in gauge theory as their Hitchin-Simpson connections are Yang-Mills instantons, i.e., critical points (in fact minima) of the Yang-Mills functional

$$\mathrm{YM}(D) = \int_X |F_D|^2.$$

Moreover, in dimension 4 they correspond to anti-self dual connections. Based on this, it is not a surprise to expect that such metrics produce examples of instantons in higher dimensions. In the context of $\mathrm{Spin}(7)-$instantons this is achieved through dimensional reduction.

Suppose that X is a Calabi-Yau $3-$fold, so that the product $X \times T^2$ admits a $\mathrm{Spin}(7)-$manifold structure given by

$$\frac{i}{2} dz \wedge d\bar{z} \wedge \omega + \frac{1}{2}\omega^2 + \mathrm{Re}(dz \wedge \Omega).$$

For a holomorphic vector bundle $\mathcal{E}$ over X and a T^2-invariant Chern connection $\tilde{\nabla}$ on the pullback $\pi^*\mathcal{E} \to X \times T^2$, we can write

$$\tilde{\nabla} = \nabla + \psi_1 ds_1 + \psi_2 ds_2,$$

where ∇ is the pullback of a Chern connection in $\pi : \mathcal{E} \to W$, ψ_1, ψ_2 the pullbacks of skew-Hermitian bundle endomorphisms and s_1, s_2 the canonical coordinates of $T^2 = S^1 \times S^1$. Thus, defining $\theta = \frac{1}{2}(\psi_1 - i\psi_2)dz$, we have

$$\tilde{\nabla} = \nabla + \theta - \theta^*.$$

Now, recall that a connection A on a vector bundle E over a $\mathrm{Spin}(7)$ −manifold (M, Ξ) is called a $\mathrm{Spin}(7)$ −instanton if

$$\star(F_A \wedge \Xi) = -F_A.$$

Thus, substituting $F_{\tilde{\nabla}}$ in the $\mathrm{Spin}(7)$ −instanton equation we obtain the following system of equations:

$$\begin{aligned}
&\star\,(F_\nabla \wedge \frac{\omega^2}{2}) = [\theta, \theta^*] \\
&\star\,(\nabla\theta \wedge \frac{\omega^2}{2}) = -\nabla\theta \\
&\star\,(\nabla\theta^* \wedge \frac{\omega^2}{2}) = -\nabla\theta^* \\
&\star\,(F_\nabla \wedge \frac{i}{2} d\bar{z} \wedge dz \wedge \omega - [\theta, \theta^*] \wedge \frac{\omega^2}{2}) = -F_\nabla.
\end{aligned}$$

It is easy to see that the first and fourth equations, as well as the second and third, are equivalent. Moreover, the second equation is equivalent to $\bar{\partial}\theta = 0$. Hence, the $\mathrm{Spin}(7)$ −instanton equation resumes to

$$\begin{aligned}
\star(F_\nabla \wedge \frac{\omega^2}{2}) &= [\theta, \theta^*] \\
\bar{\partial}\theta &= 0,
\end{aligned}$$

which is precisely the condition of $(\mathcal{E}, \theta)$ being a Higgs bundle over $X \times T^2$ admitting an Higgs-Hermite-Einstein metric (assuming $c_1(\mathcal{E}) = 0$) with Hitchin-Simpson connection ∇.

The concept of Hermite-Einstein metrics on complex vector bundles was introduced by Kobayashi in (KOBAYASHI, 1980) as a generalization of Kähler-Einstein metrics. Some time later, he showed in (KOBAYASHI, 1982) that irreducible Hermite-Einstein bundles were stable in the sense of Mumford-Takemoto. Roughly a year after, Donaldson, in (DONALDSON, 1983), established the converse for bundles over Riemann surfaces giving a new proof of the famous Narasimhan-Seshadri theorem (NARASIMHAN; SESHADRI, 1964).

This led Kobayashi and Hitchin independently to conjecture a equivalence between the moduli space of stable vector bundles and irreducible Hermite-Einstein vector bundle over a complex manifold. This conjecture, known as Hitchin-Kobayashi correspondence, would represent one of the main links between algebraic geometry and complex differential geometry.

$$\boxed{\text{Stable bundles}} \leftrightsquigarrow \boxed{\text{Irreducible HE bundles}}$$

Following the case of Riemann surfaces, the Hitchin-Kobayashi correspondence was proved by Donaldson for algebraic surfaces (DONALDSON, 1985) and later for algebraic manifolds (DONALDSON, 1987) using his method of the heat flow. Meanwhile, Uhlenbeck and Yau, using their continuity method, extended the Hitchin-Kobayashi correspondence to Kähler manifolds (UHLENBECK; YAU, 1986).

Motivated by Hitchin's previous work (HITCHIN, 1987) on the equations obtained from dimensional reduction of the self-dual Yang-Mills equations, Simpson introduced the concept of Higgs bundles in (SIMPSON, 1988) to study representations of the fundamental group of complex manifolds. After proving a Hitchin-Kobayashi type of correspondence for Higgs bundles (known as Simpson correspondence), he generalized Narasimhan-Seshadri previous work establishing a equivalence between complex representations of the fundamental group and stable Higgs bundles with trivial first Chern class.

$$\boxed{\text{Irreducible representations } \pi_1(X)} \leftrightsquigarrow \boxed{\text{Stable Higgs bundles with } c_1(E) = 0}$$

Besides the link with stable bundles, Higgs-Hermitian-Einstein bundles have found many applications within the area of gauge theory, more specifically in the study of Yang-Mills-Higgs instantons.

In (DONALDSON, 1985), Donaldson showed how to obtain examples of instantons from Hermite-Einstein metrics on holomorphic bundles with trivial first Chern class. In fact, he showed that the anti-self-dual instanton equation for integrable unitary connections was equivalent to the Hermite-Einstein equation. Later, this same kind of result would lead Hitchin in (HITCHIN, 1987) to note that, after dimensional reduction, the anti-self-dual instanton equation was equivalent to the Higgs-Hermite-Einstein equation.

More recently, in their seminal paper (DONALDSON; THOMAS, 1998), Donaldson and Thomas proposed a generalization of gauge theory to higher dimensions with a particular interest in G_2 and $Spin(7)$ manifolds. This motivated Tian's highly influential work (TIAN, 2000) where he carried out part of the proposed program studying the blow-up loci of sequence of instantons and their relation with minimal submanifolds through calibrated geometry.

Aiming to study the G_2 case, Sá Earp worked on constructing examples of G_2-instantons over Kovalev manifolds. The first step for this was developed in (EARP, 2015), where he showed the existence of HE metrics over ACyl Kähler manifolds using the heat flow method and presented a method to construct G_2-instantons from these metrics. Later, in joint work with Walpuski (EARP; WALPUSKI, 2015), they proved a gluing theorem for these instantons obtaining the desired examples.

Following Sá Earp's work, in (JACOB; WALPUSKI, 2018) Jacob and Walpuski extended his existence result on HE metrics to the case of reflexive sheaves. For this they used Uhlenbeck-Yau continuity method.

Based on this last work, I studied in this thesis a extension of their result for Higgs bundles aiming to construct $\mathrm{Spin}(7)-$instantons in a manner similar to the G_2 case explained above.

In chapter 2 we review some basic facts about Higgs bundles, Higgs-Hermite-Einstein metrics and stability. The idea is to establish the basic language that will be employed in later chapters.

In chapter 3 we introduce the concepts related to ACyl Kähler manifolds, such as asymptotically translation invariant bundles and metrics, and develop the analytical tools needed in the solution of our main theorem.

In chapter 4 we prove the main result of this thesis about the existence of HHE metrics for ATI Higgs bundles over ACyl Kähler manifolds.

Theorem 1. *Let (W, ω) be an ACyl Kähler manifold with cross-section (X, ω_X) and $(\mathcal{E}, \phi)$ a ATI Higgs bundle asymptotic to a stable Higgs bundle $(\mathcal{E}_X, \phi) \to X$. Then $(\mathcal{E}, \phi)$ admits an ATI Higgs-Hermite-Einstein metric.*

This is done using Uhlenbeck-Yau continuity method very similar to that used in (JACOB; WALPUSKI, 2018) with adaptations to the Higgs case. Assuming that the bundle is stable at infinity, we are able to find a starting metric for the continuity method. Then we compute the linearisation of the equation and use the implicit function theorem to guarantee openness of the continuity set. Finally, we prove some a priori estimates to show that the continuity set is closed.

Finally, in chapter 1.4 we show how one can obtain $\mathrm{Spin}(7)-$instantons from HHE metrics through dimensional reduction.

1 Gauge theory on Spin(7)−manifolds

In this initial chapter I will introduce the background material on $\mathrm{Spin}(7)$ −manifolds fundamental for comprehending later contents of this thesis. I will start introducing the $\mathrm{Spin}(7)$ group and some of its properties. Then I will define $\mathrm{Spin}(7)$ −manifolds and show their features. Finally I will present $\mathrm{Spin}(7)$ −instantons giving some examples and characteristics. This chapter is based on the following references (BRYANT, 1987; JOYCE, 2000; SALAMON, 1989; SALAMON; WALPUSKI, 2017).

1.1 The group Spin(7)

In Berger's famous list of possible holonomy groups the Lie group $\mathrm{Spin}(7)$ represents, along with G_2, the only exceptional cases. It is the seventh group in the *Spin* family and the first of the family that is not isomorphic to a classical Lie group. Although there are several ways of defining the *Spin* family, such as the double-cover of the *SO* family or as a subgroup of the invertible elements of the Clifford algebra, here we will adopt a more concrete definition coming from a representation into $\mathbb{R}^7$.

Definition 1. *Let $\{e^i\}_{i=1,\dots 8}$ be the canonical dual base of $\mathbb{R}^8$ and denote by $e^{i_1\cdots i_k}$ the product $= e^{i_1} \wedge \cdots \wedge e^{i_k}$. We define the standard* Cayley form *of $\mathbb{R}^8$ by*

$$\Xi_0 = \frac{1}{2}(\alpha) + \mathrm{Re}(\beta), \tag{1.1}$$

where

$$\alpha = (e^{12} + e^{34} + e^{56} + e^{78}) \quad \text{and} \quad \beta = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) \wedge (e^7 + ie^8).$$

Now we define $\mathrm{Spin}(7)$ as the stabilizer of Ξ_0 under the $\mathrm{Gl}(8,\mathbb{R})$ action:

$$\mathrm{Spin}(7) \coloneqq \{g \in \mathrm{Gl}(8,\mathbb{R}) : g^*\Xi_0 = \Xi_0\}. \tag{1.2}$$

Theorem 2 ((BRYANT, 1987, Theorem 4, p. 545))**.** *The group $\mathrm{Spin}(7)$ is a simple, compact and $1-$connected Lie group of dimension 21. Moreover $\mathrm{Spin}(7)$ is a subgroup of $\mathrm{SO}(8)$ with center $\mathbb{Z}_2$ and quotient $\mathrm{Spin}(7)/\mathbb{Z}_2 \cong \mathrm{SO}(7)$.*

It follows from the above theorem that $\mathrm{Spin}(7)$ preserves the standard orientation and the Euclidean metric of $\mathbb{R}^8$. Hence, if $\star$ denotes the Hodge star operator, we have $\star\Xi_0 = \Xi_0$, i.e., Ξ_0 is a self-dual $4-$form.

More generally, we can extend the notion of a Cayley form to abstract vector spaces using the concept of admissible forms.

Definition 2. *We say that a 4−form Ξ on a vector space V is* admissible *if there exists an isomorphism $\phi : \mathbb{R}^8 \to V$ such that $\phi^*\Xi = \Xi_0$. We denote the set of admissible 4−forms on V by $\mathcal{A}(V)$.*

If we consider the map $\mathrm{Gl}(V) \to \Lambda^4 V^*$ given by $g \mapsto g^*\Xi$, we see that it imbeds $\mathrm{Gl}(V)/\,\mathrm{Spin}(7) \simeq \mathcal{A}(V)$ as smooth submanifold of $\Lambda^4 V^*$. Moreover, since $\dim \mathrm{Gl}(V) = 64$ and $\dim \mathrm{Spin}(7) = 21$, we see that $\mathcal{A}(V)$ is 43−dimensional and has codimension 27 on $\Lambda^4 V^*$.

Proposition 1. *Let Ξ be an admissible 4−form on a vector space V. Then, the action of* Spin(7) *on $\Lambda^* V^*$ gives the following orthogonal splittings*

$$\Lambda^1 V^* = \Lambda^1_8 \qquad \Lambda^3 V^* = \Lambda^3_8 \oplus \Lambda^3_{48}$$
$$\Lambda^2 V^* = \Lambda^2_7 \oplus \Lambda^2_{21} \qquad \Lambda^4 V^* = \underbrace{\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}}_{SD} \oplus \underbrace{\Lambda^4_{35}}_{ASD}$$

where Λ^k_d denotes a d−dimensional irreducible representation of Spin(7)*. More explicitly we have*

$$\Lambda^2_7 = \{\alpha : \star(\alpha \wedge \Xi) = 3\alpha\} \qquad \Lambda^4_1 = \langle \Xi \rangle$$
$$\Lambda^2_{21} = \{\alpha : \star(\alpha \wedge \Xi) = -\alpha\} \simeq \mathfrak{spin}(7) \qquad \Lambda^4_7 = \mathfrak{so}(V) \cdot \Xi \cong \Lambda^2_7$$
$$\Lambda^3_8 = \{v \lrcorner \Xi : v \in V\} \cong \Lambda^1_8 \qquad \Lambda^4_{27} = \mathrm{Sym}_0(V) \cdot \Xi$$
$$\Lambda^3_{48} = \{\alpha : \alpha \wedge \Xi = 0\} \qquad \Lambda^4_{35} = \{\alpha : \star\alpha = -\alpha\}$$

Also note that the Hodge star gives an isometry between $\Lambda^k V^*$ and $\Lambda^{8-k} V^*$.

1.2 Spin(7)−manifolds

Let M be an 8−dimensional manifold. Using the notion of admissible 4−forms, we consider at each point $p \in M$ the set $\mathcal{A}(T_pM)$ and define the bundle $\mathcal{A}M$ as

$$\mathcal{A}M = \sqcup_{p \in M} \mathcal{A}(T_pM).$$

Although $\mathcal{A}M \subset \Lambda^4 T^*M$, it is not a vector subbundle of $\Lambda^4 T^*M$ since its fibers are isomorphic to $\mathrm{Gl}(8, \mathbb{R})/\,\mathrm{Spin}(7)$. Let $\Gamma(\mathcal{A}M)$ denote the space of smooth sections of $\mathcal{A}M$.

Definition 3. *A* Spin(7) −*structure on an 8−dimensional manifold M is a section $\Xi \in \Gamma(\mathcal{A}M)$. We call the pair (M, Ξ) an* almost Spin(7) −manifold*.*

Since an almost $\mathrm{Spin}(7)$ −manifold (M, Ξ) has an admissible form Ξ_p on T_pM for each $p \in M$, we can equip M with a canonical Riemannian metric g_Ξ and orientation vol_Ξ, and obtain a Hodge star operator $\star_\Xi$. It follows from Proposition 1 that for any almost $\mathrm{Spin}(7)$ −manifold (M, Ξ) we have a bundle decomposition of $\Lambda^k T^*M$. Using the same notation, we denote by Λ^k_d the summands and by $\pi_d : \Lambda^k T^*M \to \Lambda^k_d$ the projection.

Definition 4. *The* torsion *of a* $\mathrm{Spin}(7)$ *−structure* Ξ *is defined by the tensor*

$$\nabla_{g_\Xi}\Xi,$$

where ∇_{g_Ξ} *is the Levi-Civita connection of* g_Ξ*. We say that* Ξ *is* torsion-free *if* $\nabla_{g_\Xi}\Xi = 0$*. In this case we call* (M, Ξ) *a* $\mathrm{Spin}(7)$ *−manifold.*

It follows from the holonomy principle that for a $\mathrm{Spin}(7)$ −manifold (M, Ξ) the metric g_Ξ satisfies $\mathrm{Hol}(g_\Xi) \subset \mathrm{Spin}(7)$. Moreover, if (M, g) is a 8−dimensional Riemannian manifold with $\mathrm{Hol}(g) \subset \mathrm{Spin}(7)$ then M admits a torsion-free $\mathrm{Spin}(7)$ −structure Ξ such that $g = g_\Xi$. Hence, for 8−dimensional manifold M there is an equivalence between torsion-free $\mathrm{Spin}(7)$ −structures and $\mathrm{Spin}(7)$ holonomy metrics over it. The next theorem gives another characterization of $\mathrm{Spin}(7)$ −manifolds.

Theorem 3 ((SALAMON, 1989, Lemma 12.4, p. 176))**.** *Let* (M, Ξ) *be an almost* $\mathrm{Spin}(7)$ *−manifold, then the following are equivalent:*

- $\nabla_{g_\Xi}\Xi = 0$*;*
- $d\Xi = 0$*.*

The following results shows the main geometric properties which motivate the study of $\mathrm{Spin}(7)$ −manifolds, particularly for physicists.

Proposition 2 ((SALAMON, 1989, Corollary 12.6, p. 176))**.** *Let* (M, g) *be a Riemannian manifold with* $\mathrm{Hol}(g) \subset$ *Spin*(7)*, then* g *is Ricci flat.*

Proposition 3 ((JOYCE, 2000, Proposition 10.5.5, p.256))**.** *Let* (M, Ξ) *be an almost* $\mathrm{Spin}(7)$ *−manifold. Then* (M, g_Ξ) *is orientable and spin, with a canonical orientation and spin structure. Moreover, if* Ξ *is torsion-free, then* (M, g_Ξ) *admits a nonzero parallel positive spinor.*

In fact, if $S = S_+ \oplus S_-$ is the spin bundle of M, then there are natural isomorphisms $S_+ \cong \Lambda^0_1 \oplus \Lambda^2_7$ and $S_- \cong \Lambda^1_8$.

From Berger's list we have the following theorem which helps characterize all possible examples of $\mathrm{Spin}(7)$ −manifolds.

Theorem 4 ((JOYCE, 2000, Theorem 10.5.7, p. 256))**.** *The only non-trivial connected Lie subgroups of* $\mathrm{Spin}(7)$ *which can be holonomy group of a Riemannnian metric on an* $8-$*manifold are:*

- $\mathrm{SU}(2)$*, acting on* $\mathbb{R}^8 \simeq \mathbb{R}^4 \oplus \mathbb{C}^2$ *trivially on* $\mathbb{R}^4$ *and as usual on* $\mathbb{C}^2$*,*
- $\mathrm{SU}(2) \times \mathrm{SU}(2)$*, acting on* $\mathbb{R}^8 \simeq \mathbb{C}^2 \oplus \mathbb{C}^2$ *in the obvious way,*
- $\mathrm{SU}(3)$*, acting on* $\mathbb{R}^8 \simeq \mathbb{R}^2 \oplus \mathbb{C}^3$ *trivially on* $\mathbb{R}^2$ *and as usual on* $\mathbb{C}^3$*,*
- G_2*, acting on* $\mathbb{R}^8 \simeq \mathbb{R} \oplus \mathbb{R}^7$ *trivially on* $\mathbb{R}$ *and as usual on* $\mathbb{R}^7$*,*
- $\mathrm{Sp}(2)$*, acting as usual on* $\mathbb{R}^8 \simeq \mathbb{H}^2$*,*
- $\mathrm{SU}(4)$*, acting as usual on* $\mathbb{R}^8 \simeq \mathbb{C}^4$*.*

Therefore, if Ξ *is a torsion-free* $\mathrm{Spin}(7)-$*structure on an* $8-$*manifold, then* $\mathrm{Hol}_0(g_\Xi)$ *is one of* $\{1\}$, $\mathrm{SU}(2)$, $\mathrm{SU}(2) \times \mathrm{SU}(2)$, $\mathrm{SU}(3)$, G_2, $\mathrm{Sp}(2)$, $\mathrm{SU}(4)$ *or* $\mathrm{Spin}(7)$*.*

Thus we can use lower dimensional geometries like Calabi-Yau $2-$folds, $3-$folds or $4-$folds to obtain $\mathrm{Spin}(7)-$manifolds. Here are some explicit examples where the inclusions above produce $\mathrm{Spin}(7)-$manifolds.

Example 1. *The most elementary example of a* $\mathrm{Spin}(7)-$ *manifold, and the model to have in mind, is* $(\mathbb{R}^8, \Xi_0)$ *as its holonomy is trivial. For the compact case, one can have in mind the* $8-$*dimensional torus coming from the quotient of* $\mathbb{R}^8$ *with* $\mathrm{Spin}(7)-$*structure induced from* Ξ*.*

Example 2. *Let* (X, g, I, Ω) *be a Calabi-Yau* $4-$*fold and define the* $4-$*form*

$$\Xi = \frac{1}{2}\omega^2 + \mathrm{Re}\,\Omega.$$

Computing the local expression of Ξ *in normal coordinates one can easily check that it defines a* $\mathrm{Spin}(7)-$*structure on* X*. Moreover, since* ω *and* Ω *are* ∇_g-*parallel, it follows that* Ξ *torsion-free. Hence* (X, Ξ) *is a* $\mathrm{Spin}(7)-$*manifold with holonomy* $\mathrm{SU}(4)$*.*

Next are the main kinds of example we are interested for the construction of $\mathrm{Spin}(7)-$instantons.

Example 3. *Let* (X, g, I, Ω) *be a Calabi-Yau* $3-$*fold and* T^2 *the torus with complex structure coming from the lattice* $\mathbb{C}/\Lambda(1, i)$*. If we define*

$$\Xi = \frac{i}{2}dz \wedge d\bar{z} \wedge \omega + \frac{1}{2}\omega^2 + \mathrm{Re}(dz \wedge \Omega),$$

the same arguments used in the previous example show that Ξ *defines a torsion-free* $\mathrm{Spin}(7)-$*structure on the* $X \times T^2$*.*

All the above examples had holonomy strictly contained in Spin(7). For examples of holonomy equal to Spin(7) the following theorem gives a topological criterion.

Theorem 5 ((JOYCE, 2000, Theorem 10.6.1, p. 259))**.** *Let* (M, Ξ) *be a compact* Spin(7) − *manifold. Then the* $\hat{\mathrm{A}}-$*genus* $\hat{\mathrm{A}}(M)$ *of* M *satisfies*

$$24\hat{\mathrm{A}}(M) = -1 + b^1 - b^2 + b^3 + b^4_+ - 2b^4_-, \tag{1.3}$$

where b^i *are the Betti numbers on* M *and* $b^4_\pm$ *the dimensions of* $H^4_\pm(M, \mathbb{R})$*. Moreover, if* M *is* 1−*connected then* $\hat{\mathrm{A}}(M)$ *is 1, 2, 3 or 4, and the holonomy group* $\mathrm{Hol}(g)$ *of* g *is determined by* $\hat{\mathrm{A}}(M)$ *as follows:*

- $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *if and only if* $\hat{\mathrm{A}}(M) = 1$*,*
- $\mathrm{Hol}(g) = \mathrm{SU}(4)$ *if and only if* $\hat{\mathrm{A}}(M) = 2$*,*
- $\mathrm{Hol}(g) = \mathrm{Sp}(2)$ *if and only if* $\hat{\mathrm{A}}(M) = 3$*,*
- $\mathrm{Hol}(g) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ *if and only if* $\hat{\mathrm{A}}(M) = 4$*.*

Examples with holonomy equal to Spin(7) are much more dificult to obtain and every compact example obtained so far involves some kind of gluing construction with no explicit description of the metrics. The first examples were obtained by Bryant in (BRYANT, 1987) using the theory of exterior differential systems. Later, Bryant and Salamon in (BRYANT; SALAMON, 1989) obtained explicit examples with complete metrics on noncompact manifolds. In the compact case Joyce in (JOYCE, 1996; JOYCE, 1999) constructed examples by resolving singularities of orbifolds with torsion-free Spin(7) −structures.

1.3 Spin(7)−instantons

Let (M, Ξ) be a Spin(7) −manifold and G a semi-simple compact Lie group. If E is a G−bundle over M we will denote by $\mathcal{A}(E)$ the space of connections on E and by $\mathfrak{g}_E$ the associated adjoint bundle.

Definition 5. *A connection* $A \in \mathcal{A}(E)$ *on* E *is called a* Spin(7) −*instanton if it satisfies*

$$\star(F_A \wedge \Xi) = -F_A,$$

which means $\pi_7(F_A) = 0$*.*

Spin(7) −instantons were first discussed in the physics literature by various authors (CORRIGAN et al., 1983; WARD, 1984) and later presented to a broader mathematical audience by Donaldson-Thomas (DONALDSON; THOMAS, 1998, Section 3). The first thorough study of these instantons can be found in Lewis's thesis (LEWIS, 1999); there he proved a variety of elementary results, studied the bubbling of a family of instantons and constructed a non-trivial example over a manifold with holonomy Spin(7). More recently, Tanaka (TANAKA, 2012) exhibited a method to construct Spin(7) −instantons on Spin(7) −manifolds obtained by Joyce (JOYCE, 1999).

Example 4. *The most trivial example of* Spin(7) *−instantons are flat connections.*

Example 5. *Let* (X, ω, Ω) *be a Calabi-Yau* $4-$*fold,* E *a* $G-$*bundle over* X *and* A *a* Hermitian-Yang-Mills connection*, i.e.,*

$$\Lambda F_A = 0 \quad \text{and} \quad \mathrm{F}_{\mathrm{A}}^{0,2} = 0$$

or equivalently

$$F_A \wedge \omega^3 = 0 \quad \text{and} \quad \mathrm{F}_{\mathrm{A}} \wedge \mathrm{Re}\,\Omega = 0.$$

If we take the Spin(7) *−structure* $\Xi = \frac{1}{2}\omega^2 + \mathrm{Re}\,\Omega$ *as in Example 2, we can compute*

$$\begin{aligned} \star(F_A \wedge \Xi) &= \star(\frac{1}{2}F_A \wedge \omega^2 + F_A \wedge \mathrm{Re}\,\Omega) \\ &= \frac{1}{2} \star (F_A \wedge \omega) \end{aligned}$$

Example 6. *The Levi-Civita connection of a* Spin(7) *−manifold* (M, Ξ) *is a* Spin(7) *−instanton. To see this note that, in the case of* $\mathrm{Hol}(g) \subset \mathrm{Spin}(7)$*, the Riemannian curvature tensor* R *is an element of* $S^2\mathfrak{spin}(7)$ *at each point. Hence, from the decomposition in Proposition 1, it follows that* $\star(R \wedge \Xi) = -R$*.*

1.4 Constructing Spin(7) −instantons

In this last section 1.4 we show how HHE metrics are related to $Spin(7)-$instantons using dimensional reduction. Following the same ideas in (HITCHIN, 1987) we are able to show that HHE metric are solution to the $Spin(7)-$instanton equation for bundles with trivial first Chern class. Here we will also explain how this may be used to obtain $Spin(7)-$instanton over compact manifolds following a twisted connected sum construction similar to the G_2 case.

1.4.1 Spin(7)−instantons and dimensional reduction

Let $(X \times T^2, \Xi)$ be the Spin(7) −manifold of Example 3 and take $\mathcal{E}$ to be a holomorphic vector bundle over X. If we assume that $\tilde{\nabla}$ is a T^2-invariant integrable unitary connection on the pullback $\pi^*\mathcal{E} \to X \times \overline{T^2}$, we can write

$$\tilde{\nabla} = \nabla + \psi_1 ds_1 + \psi_2 ds_2,$$

where ∇ is the pullback of an integrable unitary connection in $\pi : \mathcal{E} \to W$, ψ_1, ψ_2 the pullbacks of skew-Hermitian bundle endomorphisms and s_1, s_2 the canonical coordinates of $T^2 = S^1 \times S^1$. Taking $\theta = \frac{1}{2}(\psi_1 - i\psi_2)dz$, we can write

$$\tilde{\nabla} = \nabla + \theta - \theta^*.$$

Consider now the Spin(7) −instanton equation for $\tilde{\nabla}$. Since $F_{\tilde{\nabla}} = F_\nabla + \nabla\theta - \nabla\theta^* - [\theta, \theta^*]$, after expanding the equation and using the degrees compatibility we obtain the following system of equations:

$$\begin{aligned}
&\star\,(F_\nabla \wedge \frac{\omega^2}{2}) = [\theta, \theta^*]\\
&\star\,(\nabla\theta \wedge \frac{\omega^2}{2}) = -\nabla\theta\\
&\star\,(\nabla\theta^* \wedge \frac{\omega^2}{2}) = -\nabla\theta^*\\
&\star\,(F_\nabla \wedge \frac{i}{2}d\bar{z} \wedge dz \wedge \omega - [\theta, \theta^*] \wedge \frac{\omega^2}{2}) = -F_\nabla.
\end{aligned}$$

It is easy to see that the first and fourth equations, as well as the second and third, are equivalent. Moreover, the second equation is equivalent to $\bar{\partial}\theta = 0$. Hence, the Spin(7) −instanton equation resumes to

$$\begin{aligned}
\star(F_\nabla \wedge \frac{\omega^2}{2}) &= [\theta, \theta^*]\\
\bar{\partial}\theta &= 0.
\end{aligned}$$

This is precisely the condition of $(\pi^*\mathcal{E}, \theta)$ being a Higgs bundle over $W \times T^2$ admitting an Higgs-Hermite-Einstein metric (assuming $c_1(\mathcal{E}) = 0$).

Thus if we start with a Higgs bundle with θ of the form $\theta = \frac{1}{2}(\psi_1 - i\psi_2)dz$ and satisfying the above equations, then we can produce a Spin(7) −instanton out of it.

1.4.2 Twisted connected sums for G_2 −manifolds

As explained above, by solving the PHYM problem for $(\mathcal{E}, \theta)$ over W we provide a T^2−invariant $Spin(7)$−instanton over $W \times \overline{T^2}$. Now what we expect to do in the future is to glue a pair of such instantons along a gluing of the base spaces obtaining a $Spin(7)$−instanton over a compact $Spin(7)$−manifold.

A *building block* (X, Σ) consists of a smooth projective 3−fold X and a smooth anticanonical $K3$ surface Σ with trivial normal bundle. Given such pair and a hyperkähler structure on Σ we can make $W := X \backslash \Sigma$ into an asymptotically cylindrical Calabi-Yau 3−fold with end modelled on $(0, \infty) \times S^1 \times \Sigma$.

Given a pair of building blocks $(X_\pm, \Sigma_\pm)$, a *matching data* $\mathfrak{m} = \{(\omega_{I,\pm}, \omega_{J,\pm}, \omega_{K,\pm}), \mathfrak{r}\}$ is a set consisting of two hyperkähler structures $(\omega_{I,\pm}, \omega_{J,\pm}, \omega_{K,\pm})$, one for each $\Sigma_\pm$, and diffeomorphism $\mathfrak{r} : \Sigma_+ \to \Sigma_-$, called a *hyperkähler rotation*, such that $[\omega_{I,\pm}]$ is the restriction of the Kähler class on $X_\pm$ to $\Sigma_\pm$ and $\mathfrak{r}$ satisfies

$$\mathfrak{r}^*\omega_{I,-} = \omega_{J,+} \quad \mathfrak{r}^*\omega_{J,-} = \omega_{I,+} \quad \mathfrak{r}^*\omega_{K,-} = -\omega_{K,+}.$$

Using a matching data $\mathfrak{m}$ for a pair of building blocks $(X_\pm, \Sigma_\pm)$, Kovalev and Lee (KOVALEV; LEE, 2011) showed that one can glue the Acyl manifolds $W_+ \times S^1$, $W_- \times S^1$ along the ends, as depicted below, obtaining a compact $7-$dimensional manifold Y with a family of torsion-free $G_2)-$structures $(\phi_t)_{t \geqslant t_0}$. This manifold with the family of G_2-structures is called the *twisted connected sum* of the pair $(X_\pm, \Sigma_\pm)$.

$$\begin{array}{ccc}
\Sigma_+ & \overset{\mathfrak{r}}{\text{———}} & \Sigma_- \\
\times & & \times \\
S^1 & & S^1 \\
\times & \times & \times \\
S^1 & & S^1
\end{array}$$

Based on this gluing, Sá Earp and Walpuski (EARP; WALPUSKI, 2015) showed how to construct G_2-instantons over twisted connected sums, with a concrete example later being presented by Walpuski (WALPUSKI, 2016).

1.4.3 Spin(7)−instantons over twisted connected sums

Following the twisted connected sum idea, I propose one should search for an analogous construction for $Spin(7)-$manifolds. Using the same notation, consider the pairs $W_+ \times T^2$, $W_- \times T^2$. Then if we glue their ends following Kovalev and Lee's method, but using a diffeomorphism $f : T^3 \to T^3$ that acts trivially on the homotopy groups, we may expect to obtain a compact $8-$dimensional manifold with holonomy contained in $Spin(7)$.

Although this may not be a holonomy $Spin(7)$ manifolds for every pair of building blocks like in the G_2 case, we hope to obtain, by imposing conditions on the building blocks, some examples in which the holonomy is indeed $Spin(7)$.

For this kind of examples, we can apply a method similar to that of Sá Earp and Walpuski using my result for Higgs bundles. Therefore we obtain a construction for $Spin(7)-$instantons over these twisted connected sums

2 Higgs bundles

In this chapter I will briefly discuss Higgs bundles, Higgs-Hermite-Einstein metrics and stability. The idea is to review the basic definitions and results that will appear in later chapters. The main references for this chapter are (SIMPSON, 1988) and (SIMPSON, 1992). In the last section, for the approach on stability, I will follow (KOBAYASHI, 1987).

2.1 Higgs sheaves and stability

Let X be a $n-$dimensional complex manifold and $\mathcal{O}_X$ its structure sheaf. We will call a *sheaf* over X any sheaf of $\mathcal{O}_X-$modules over the ringed space $(X, \mathcal{O}_X)$.

Definition 6. *A* Higgs sheaf *over X is a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is coherent sheaf over X and $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ is a sheaf morphism, called* Higgs field*, such that the composite $\theta \wedge \theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^2_X$ vanishes.*

In this context, we say that a Higgs sheaf $(\mathcal{E}, \theta)$ is *torsion free* (resp. *reflexive*, *locally free*), if $\mathcal{E}$ is torsion free (resp. reflexive, locally free). Moreover, as we can identify locally-free sheaves with holomorphic vector bundles, we will refer to a locally free Higgs sheaf as a Higgs bundle.

Definition 7. *A* morphism *between Higgs sheaves $(\mathcal{E}_1, \theta_1)$ and $(\mathcal{E}_2, \theta_2)$ over X is a sheaf morphism $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ such that the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r, "\phi"] \arrow[d, "\theta_1"'] & \mathcal{E}_2 \arrow[d, "\theta_2"] \\
\mathcal{E}_1 \otimes \Omega^1_X \arrow[r, "\phi\otimes \mathrm{Id}"] & \mathcal{E}_2 \otimes \Omega^1_X
\end{tikzcd}$$

Based on this definition, a *Higgs subsheaf* of $(\mathcal{E}, \theta)$ is a pair $(\mathcal{F}, \theta|_{\mathcal{F}})$ where $\mathcal{F}$ is a subsheaf of $\mathcal{E}$ satisfying $\theta(\mathcal{F}) \subset \mathcal{F} \otimes \Omega^1 X$. A *Higgs quotient* of $(\mathcal{E}, \theta)$ is a pair $(\mathcal{Q}, \theta|_{\mathcal{Q}})$ where $\mathcal{Q}$ is a quotient sheaf of $\mathcal{E}$ whose kernel is a Higgs subsheaf.

Note that if $\phi : (\mathcal{E}_1, \theta_1) \to (\mathcal{E}_2, \theta_2)$ is a morphism, then it follows from the commutativity property that $(\ker(\phi), \theta_1|_{\ker(\phi)})$ is a Higgs subsheaf of $(\mathcal{E}_1, \theta_1)$ and $(\mathrm{im}(\phi), \theta_2|_{\mathrm{im}(\phi)})$ is a Higgs subsheaf of $(\mathcal{E}_2, \theta_2)$. Thus, $(\mathrm{im}(\phi), \theta_2|_{\mathrm{im}(\phi)})$ is a Higgs quotient of $(\mathcal{E}_1, \theta_1)$ and $(\mathcal{E}_2/\,\mathrm{im}(\phi), \theta_2|_{\mathcal{E}_2/\,\mathrm{im}(\phi)})$ is a Higgs quotient of $(\mathcal{E}_2, \theta_2)$.

Definition 8. *We say that a Higgs sheaf $\mathcal{E}$ is* simple *if every morphism $\phi : (\mathcal{E}, \theta) \to (\mathcal{E}, \theta)$ is a scalar multiple of the identity map.*

Now, assume that X is compact and admits a Kähler form ω. If $(\mathcal{E},\theta)$ is a Higgs sheaf over X we define the $\omega-$*degree* of $(\mathcal{E},\theta)$ as

$$\deg_\omega(\mathcal{E},\theta) = \int_X c_1(\mathcal{E}) \wedge \omega^{n-1}$$

and the rank of $(\mathcal{E},\theta)$ as

$$\mathrm{rk}(\mathcal{E},\theta) = \mathrm{rk}\,\mathcal{E}.$$

In the case where $\mathrm{rk}(\mathcal{E},\theta) > 0$ we define the $\omega-$*slope* of $(\mathcal{E},\theta)$ to be

$$\mu_\omega(\mathcal{E},\theta) = \frac{\deg_\omega(\mathcal{E},\theta)}{\mathrm{rk}(\mathcal{E},\theta)}.$$

When working over a compact Kähler manifold (X,ω) we will always use ω for the concepts above, so that we will omit it for simplicity.

Definition 9. *A Higgs sheaf $(\mathcal{E},\theta)$ over a Kähler manifold (X,ω) is called* stable *(resp.* semistable*), if it is torsion free and for any Higgs subsheaf $(\mathcal{F},\theta|_\mathcal{F})$, with $0 < \mathrm{rk}(\mathcal{F},\theta|_\mathcal{F}) < \mathrm{rk}(\mathcal{E},\theta)$, the inequality*

$$\mu(\mathcal{F},\theta|_\mathcal{F}) < \mu(\mathcal{E},\theta) \quad (\text{resp.} \quad \mu(\mathcal{F},\theta|_\mathcal{F}) \leqslant \mu(\mathcal{E},\theta))$$

holds.

The following results generalize to the case of Higgs sheaves the standard results on (semi-)stability of analytic sheaves. Since the strategy of the proof is the same, we refer the reader to (KOBAYASHI, 1987, Section 5.7).

Lemma 1. *If*

$$0 \to (\mathcal{E}_1,\theta_1) \to (\mathcal{E}_2,\theta_2) \to (\mathcal{E}_3,\theta_3) \to 0$$

is an exact sequence of Higgs sheaves over a compact Kähler manifold (X,ω), then

$$r_1(\mu(\mathcal{E}_2,\theta_2) - \mu(\mathcal{E}_1,\theta_1)) + r_3(\mu(\mathcal{E}_2,\theta_2) - \mu(\mathcal{E}_3,\theta_3)) = 0,$$

where $r_1 = \mathrm{rk}(\mathcal{E}_1,\theta_1)$ and $r_3 = \mathrm{rk}(\mathcal{E}_3,\theta_3)$.

Corollary 1. *Let $(\mathcal{E},\theta)$ be a torsion free Higgs sheaf over a compact Kähler manifold (X,ω). Then $(\mathcal{E},\theta)$ is stable (resp. semistable) if and only if for any Higgs quotient $(\mathcal{Q},\theta|_\mathcal{Q})$, with $0 < \mathrm{rk}(\mathcal{E},\theta) < \mathrm{rk}(\mathcal{Q},\theta|_\mathcal{Q})$, the inequality*

$$\mu(\mathcal{E},\theta) < \mu(\mathcal{Q},\theta|_\mathcal{Q}) \quad (\text{resp.} \quad \mu(\mathcal{E},\theta) \leqslant \mu(\mathcal{Q},\theta|_\mathcal{Q}))$$

holds.

Proposition 4. *Let $(\mathcal{E}_1,\theta_1)$ and $(\mathcal{E}_2,\theta_2)$ be semistable Higgs sheaves over a compact Kähler manifold (X,ω). Let $\phi : (\mathcal{E}_1,\theta_1) \to (\mathcal{E}_2,\theta_2)$ be a morphism.*

1. *If* $\mu(\mathcal{E}_1, \theta_1) > \mu(\mathcal{E}_2, \theta_2)$*, then* $\phi = 0$*;*

2. *If* $\mu(\mathcal{E}_1, \theta_1) = \mu(\mathcal{E}_2, \theta_2)$ *and if* $(\mathcal{E}_1, \theta_1)$ *is stable, then* $\mathrm{rk}(\mathcal{E}_1, \theta_1) = \mathrm{rk}(\phi(\mathcal{E}_1, \theta_1))$ *and* ϕ *is injective unless* $\phi = 0$*;*

3. *If* $\mu(\mathcal{E}_1, \theta_1) = \mu(\mathcal{E}_2, \theta_2)$ *and if* $(\mathcal{E}_2, \theta_2)$ *is stable, then* $\mathrm{rk}(\mathcal{E}_2, \theta_2) = \mathrm{rk}(\phi(\mathcal{E}_1, \theta_1))$ *and* ϕ *is generically surjective unless* $\phi = 0$.

Corollary 2. *Let* $(\mathcal{E}_1, \theta_1)$ *and* $(\mathcal{E}_2, \theta_2)$ *be semistable Higgs bundles over a compact Kähler manifold* (X, ω) *such that* $\mathrm{rk}(\mathcal{E}_1, \theta_1) = \mathrm{rk}(\mathcal{E}_2, \theta_2)$ *and* $\deg(\mathcal{E}_1, \theta_1) = \deg(\mathcal{E}_2, \theta_2)$*. If* $(\mathcal{E}_1, \theta_1)$ *or* $(\mathcal{E}_2, \theta_2)$ *is stable, then any nonzero morphism* $\phi : (\mathcal{E}_1, \theta_1) \to (\mathcal{E}_2, \theta_2)$ *is an isomorphism.*

Corollary 3. *Every stable Higgs bundle* $(\mathcal{E}, \theta)$ *over a compact Kähler manifold* (X, ω) *is simple.*

2.2 Hermitian Higgs bundles

In this section we will introduce the notion of a Higgs bundle and present some results involving Hermitian metrics on them. For this we will denote by X a $n-$dimensional complex manifold.

Similar to the case of holomorphic vector bundles, in a Higgs bundle $(\mathcal{E}, \theta)$, there is a natural differential operator associated to it given by $D'' = \bar{\partial} + \theta$. Moreover, since θ is holomorphic and $\bar{\partial}^2 = \theta^2 = 0$, it easily follows that

$$(D'')^2 = \bar{\partial}^2 + \bar{\partial}(\theta) + \theta^2 = 0.$$

Motivated by this, we define the following class of differential operators that generalize partial connections.

Definition 10. *Let E be a complex vector bundle over a complex manifold X. A* partial Higgs connection *on E is a linear operator* $D'' : \Gamma(E) \to \Gamma(E) \otimes \Omega^1 X$ *which satisfies*

$$D''(fs) = (\bar{\partial} f)s + f(D''s)$$

for all $f \in C^\infty(X)$ *and* $s \in \Gamma(E)$*.*

It follows from the definition above that the components of a partial Higgs connection D'' satisfies

$$(D'')^{1,0}(fs) = f((D'')^{1,0}s) \quad \text{and} \quad (D'')^{0,1}(fs) = (\bar{\partial} f)s + f((D'')^{0,1}s)$$

for all $f \in C^\infty(X)$ and $s \in \Gamma(E)$. Thus, we have

$$D'' = \bar{\partial} + \theta \tag{2.1}$$

where $\theta \in \Omega^{1,0}(E)$ constitutes the $(1,0)-$component of D'' and $\bar{\partial}$ is a partial connection constituting the $(0,1)-$component of D''. As for covariant exterior derivatives, we can extend D'' to $\Omega^{p,q}(E)$ using the partial Leibniz rule, and see that $(D'')^2$ is a tensor that can be identified as an element of $\Omega^2(\operatorname{End} E)$. We call a partial Higgs connection integrable if it comes from a Higgs bundle structure on E, and like in the case of holomorphic bundles, we have the following result for integrability of partial Higgs connections.

Proposition 5. *A partial Higgs connection D'' on a complex vector bundle E over a complex manifold X is integrable if and only if $(D'')^2 = 0$.*

Proof. As we have seen above, $(D'')^2$ always vanishes for integrable partial Higgs connections. On the other hand, if $(D'')^2 = 0$, then by (2.1) we have

$$0 = (D'')^2 = (\bar{\partial})^2 + \bar{\partial}(\theta) + \theta^2.$$

Since each term of the sum has bidegree $(0,2)$, $(1,1)$ and $(2,0)$ respectively it follows that

$$(\bar{\partial})^2 = 0 \quad \bar{\partial}(\theta) = 0 \quad \theta^2 = 0.$$

These are precisely the conditions for $\bar{\partial}$ to define a holomorphic structure on E such that θ is a Higgs field, hence D'' is integrable. □

Hence, we see from the result above that the partial Higgs connection D'' associated to a Higgs bundle $(\mathcal{E}, \theta)$ carry all its information, in the same manner a partial connection $\bar{\partial}$ carry all the information of a holomorphic bundle $\mathcal{F}$.

Definition 11. *A* map *between Higgs bundles $(\mathcal{E}_1, \theta_1)$ and $(\mathcal{E}_2, \theta_2)$ over X is a holomorphic vector bundle map $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ such that the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r, "\phi"] \arrow[d, "\theta_1"'] & \mathcal{E}_2 \arrow[d, "\theta_2"] \\
\mathcal{E}_1 \otimes \Omega^1_X \arrow[r, "\phi \otimes \mathrm{Id}"] & \mathcal{E}_2 \otimes \Omega^1_X
\end{tikzcd}$$

If $(\mathcal{E}_1, \theta_1)$ and $(\mathcal{E}_2, \theta_2)$ are Higgs bundles, then we have a natural Higgs field θ_{12} on $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ defined by

$$(\theta_{12}\phi)(s) = (\phi \otimes \mathrm{Id})(\theta_1(s)) + \theta_2(\phi(s)),$$

for all $\phi \in \Gamma(\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_2))$ and $s \in \Gamma(E_1)$. This, on the other hand, induces a partial Higgs connection on $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ given by

$$(D''_{12}\phi)(s) = (\phi \otimes \mathrm{Id})(D''_1(s)) + D''_2(\phi(s)),$$

for all $\phi \in \Gamma(\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2))$ and $s \in \Gamma(E_1)$. Using the bidegree decomposition, we have that

$$D''_{12}\phi = 0 \iff \bar{\partial}_{12}\phi = 0 \quad \text{and} \quad \theta_{12}\phi = 0,$$

where $\bar{\partial}_{12}$ is the partial connection associated to the holomorphic structure of $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$. Hence, we conclude that $\phi \in \Gamma(\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2))$ is a Higgs bundle map if and only if $\phi \in \ker D''_{12}$.

Recall that a Higgs bundle $(\mathcal{E}, \theta)$ is *simple* if every Higgs endomorphism of $(\mathcal{E}, \theta)$ is a homotety. Following the discussion above, one has the following corollary.

Corollary 4. *A Higgs bundle $(\mathcal{E}, \theta)$ is simple if and only if* $\ker D''_{\mathrm{End}\,\mathcal{E}} = \mathbb{C}$.

As we will see later, this result have a nice analytical implication once we provide the Higgs bundle with a Hermitian metric.

Let $(\mathcal{E}, \theta)$ be a Higgs bundle over a complex manifold X. If H is an Hermitian metric on E we call the triple $(\mathcal{E}, \theta, H)$ a *Hermitian Higgs bundle* over X. Similar to the case of holomorphic vector bundles, there is a natural connection involving the holomorphic structure $\bar{\partial}$, the Higgs bundle θ and the Hermitian metric H associated with $(\mathcal{E}, \theta, H)$. The *Hitchin-Simpson connection* of $(\mathcal{E}, \theta, H)$ is defined as

$$D_H = \nabla_H + \theta + \theta^*, \tag{2.2}$$

where ∇_H is the Chern connection of $(\mathcal{E}, H)$ and θ^* satisfies

$$\langle \theta s, t \rangle_H = \langle s, \theta^* t \rangle_H.$$

Note that θ^* depends on the metric H and, since $\bar{\partial}\theta = 0$, we have $\partial_H \theta^* = 0$.

Let F_H denote the curvature of the Hitchin-Simpson connection D_H and F_{∇_H} the curvature of the Chern connection ∇_H. Using (2.2) we have that

$$\begin{aligned} F_H = D_H^2 &= (\nabla_H + \theta + \theta^*)(\nabla_H + \theta + \theta^*) \\ &= F_{\nabla_H} + \theta\nabla_H + \nabla_H\theta + \theta^*\nabla_H + \nabla_H\theta^* \\ &+ \theta \wedge \theta^* + \theta^* \wedge \theta, \end{aligned}$$

where in the last equation we use that $\theta^2 = (\theta^*)^2 = 0$. Since $[\theta, \theta^*] = \theta \wedge \theta^* + \theta^* \wedge \theta$ and

$$\partial_H(\theta) = \theta\nabla_H + \nabla_H\theta \quad \text{and} \quad \bar{\partial}_H(\theta^*) = \theta^*\nabla_H + \nabla_H\theta^*,$$

we see that the curvature of D_H can be written as

$$F_H = F_{\nabla_H} + \partial_H(\theta) + \bar{\partial}_H(\theta^*) + [\theta, \theta^*]. \tag{2.3}$$

Moreover, considering that $\partial_H(\theta)$ is of type $(2,0)$ and $\bar{\partial}_H(\theta^*)$ is of type $(0,2)$, we obtain that

$$F_H^{1,1} = F_{\nabla_H} + [\theta, \theta^*]. \tag{2.4}$$

Observe that we can decompose $D_H = D'_H + D''$, where

$$D'_H = \partial_H + \theta^* \quad \text{and} \quad D'' = \bar{\partial} + \theta. \tag{2.5}$$

Thus, like the Chern connection, one can isolate all the dependence of D_H on H to the operator D'_H, so that D'' remains unchanged. Besides, since θ is holomorphic and $\theta^2 = 0$, it follows that $(D'_H)^2 = (D'')^2 = 0$, and we have

$$F_H = D'_H D'' + D'' D'_H \tag{2.6}$$

which is similar to case of Chern connection.

Now, assume that X admits a Kähler form ω, so that (X, ω) is a Kähler manifold, and denote by Λ the contraction operator dual to the Lefschetz operator. Applying $\mathrm{i}\,\Lambda$ to F_H and using (2.4), we see that the *mean curvature* of D_H is given by

$$K_H := \mathrm{i}\,\Lambda F_H = K_{\nabla_H} + \mathrm{i}\,\Lambda[\theta, \theta^*],$$

where $K_{\nabla_H} = i\Lambda F_{\nabla_H}$ is the mean curvature of the Chern connection ∇_H. Moreover, since K_{∇_H} is Hermitian and

$$(\mathrm{i}\,\Lambda[\theta, \theta^*])^* = -\,\mathrm{i}\,\Lambda(-[\theta, \theta^*]) = \mathrm{i}\,\Lambda[\theta, \theta^*],$$

we conclude that K_H is a Hermitian endomorphism of (E, H).

Definition 12. *Let $(\mathcal{E}, \theta)$ be a Higgs bundle over a Kähler manifold (X, ω) and H a Hermitian metric on it. We say that H is* weak Higgs-Hermite-Einstein (wHHE) *if the trace-free part of the mean curvature of D_H, given by*

$$\tilde{K}_H := K_H - \frac{\mathrm{tr}(K_H)}{\mathrm{rk}\,\mathcal{E}} \cdot \mathrm{Id}_{\mathcal{E}},$$

vanishes. Moreover, if $\mathrm{tr}(K_H)$ is constant we say that H is a Higgs-Hemite-Einstein *(HHE) metric.*

The following result, which was first presented in (SIMPSON, 1988, Lemma 3.1(a)), generalize the Kähler identities for Higgs bundles and is the main reason to set the Hitchin-Simpson connection as we defined.

Lemma 2. *Let $(\mathcal{E}, \theta, H)$ be a Hermitian Higgs bundle over a Kähler manifold (X, ω). Then, the Hitchin-Simpson connection D_H satisfies*

$$(D'_H)^\dagger = \mathrm{i}[\Lambda, D''] \quad \text{and} \quad (D'')^\dagger = -\,\mathrm{i}[\Lambda, D'_H],$$

where $(D'_H)^\dagger := -\bar{\star} D'_H \bar{\star}$ and $(D'')^\dagger := \bar{\star} D'' \bar{\star}$. When X is compact $(D'_H)^\dagger$ and $(D'')^\dagger$ are the formal L^2-adjoints of D'_H and D''.

Proof. Consider normal coordinates $(z_1, \dots, z_n)$ and an unitary frame $\{e_1, \dots, e_k\}$ around $x \in X$. In this setting we have

$$\mathrm{i}[\Lambda, dz^i] = (d\bar{z}^i)^\dagger \quad \text{and} \quad -\mathrm{i}[\Lambda, d\bar{z}^i] = (dz^i)^\dagger,$$

because $\omega = \frac{\mathrm{i}}{2}(\sum_{i=1}^{n} dz^i \wedge d\bar{z}^i)$. Thus, using that $\theta = \sum A_i dz^i$ and $\theta^* = \sum A_j^* d\bar{z}^i$ for matrix-valued functions A_i, we obtain

$$\mathrm{i}[\Lambda, \theta] = \sum A_i^* \,\mathrm{i}[\Lambda, dz^i] = (\theta^*)^\dagger \quad \text{and} \quad -\mathrm{i}[\Lambda, \theta^*] = -\sum A_i \,\mathrm{i}[\Lambda, d\bar{z}^i] = (\theta)^\dagger.$$

By the standard Kähler identities, we know

$$(\partial_H)^\dagger = \mathrm{i}[\Lambda, \bar{\partial}] \quad \text{and} \quad (\bar{\partial})^\dagger = -\mathrm{i}[\Lambda, \partial_H],$$

hence using linearity we can sum both set of equations above and obtain our result. □

It follows from the lemma above that

$$\begin{aligned}\Delta_{D_H} &:= D_H^\dagger D_H = ((D'_H)^\dagger + (D'')^\dagger)(D'_H + D'') = \mathrm{i}\,\Lambda D'' D'_H - \mathrm{i}\,\Lambda D'_H D'' \\ &= (D'_H)^\dagger (D'_H) + (D'')^\dagger (D'') = \Delta_{D'_H} + \Delta_{D''}.\end{aligned}$$

Moreover, we have from (2.6)

$$K_H = \mathrm{i}\,\Lambda(D'_H D'' + D'' D'_H) = \Delta_{D'_H} - \Delta_{D''}.$$

Hence we obtain that

$$\Delta_{D'_H} = \frac{1}{2}\Delta_{D_H} + \frac{1}{2}K_H \quad \text{and} \quad \Delta_{D''} = \frac{1}{2}\Delta_{D_H} - \frac{1}{2}K_H. \tag{2.7}$$

If we still denote by D_H the connection induced on $\operatorname{End}\mathcal{E}$, we have from (2.7) that

$$\Delta_{D'_H} = \frac{1}{2}\Delta_{D_H} + \frac{1}{2}[K_H, \cdot] \quad \text{and} \quad \Delta_{D''} = \frac{1}{2}\Delta_{D_H} - \frac{1}{2}[K_H, \cdot]. \tag{2.8}$$

Therefore, if H is weak Higgs-Hermite-Einstein, we see that both operators coincide.

Now, let H_1, H_2 be Hermitian metrics on a Higgs Higgs $(\mathcal{E}, \theta)$ and denote by $H_1^{-1}H_2$ the positive Hermitian endomorphism satisfying

$$H_2(u, v) = H_1(H_1^{-1}H_2 u, v) \tag{2.9}$$

for all $u, v \in E$. Since the exponential maps diffeomorphically the set of Hermitian matrices to the set of positive Hermitian matrices, we know there is $s \in C^\infty(X, \mathrm{i}\,\mathfrak{su}(E, H))$ such that $\mathrm{e}^s = H_1^{-1}H_2$. Thus computing how the Hitchin-Simpson connection change when associated with H_2 we have

$$\begin{aligned}D'_{H_2} &= \partial_{H_2} + \theta^{*H_2} = \partial_{H_1} + \mathrm{e}^{-s}\,\partial_{H_1}(\mathrm{e}^s) + \mathrm{e}^{-s}\,\theta^{*H_1}\,\mathrm{e}^s \\ &= D'_{H_1} + \mathrm{e}^{-s}\,D'_{H_1}(\mathrm{e}^s).\end{aligned} \tag{2.10}$$

Using (2.6) and applying $\mathrm{i}\,\Lambda$ to it we obtain

$$K_{H_2} = K_{H_1} + \mathrm{i}\,\Lambda\big(\mathrm{e}^{-s}\,D'_{H_1}(\mathrm{e}^s)D'' + D''\,\mathrm{e}^{-s}\,D'_{H_1}(\mathrm{e}^s)\big) = K_{H_1} + D''\big(\mathrm{e}^{-s}\,D'_{H_1}(\mathrm{e}^s)\big). \quad (2.11)$$

To end this section we will show some interesting results where the theory for holomorphic extends to the case of Higgs bundles.

Lemma 3. *Let $(\mathcal{E},\theta,H)$ be a Hermitian Higgs bundle over a compact Kähler manifold (X,ω). If $(\mathcal{E},\theta)$ is simple then there is a constant $C = C(\omega,\mathcal{E},\theta,H) > 0$ such that*

$$\int_X |s|^2 \leqslant C\int_X |D''s|^2,$$

for every trace-free $s \in \Gamma(\operatorname{End} E)$.

Proof. For any $s \in \Gamma(\operatorname{End} E)$ we have

$$\int_X |D''s|^2 = \int_X \langle \Delta_{D''}s, s\rangle.$$

Thus, using that $(\mathcal{E},\theta)$ is simple, we see that $\ker \Delta_{D''} = \ker D'' = \{\mathbb{C}\cdot \mathrm{Id}\}$. Since $\Delta_{D''}$ is selfadjoint and elliptic, we know that it has a nonnegative discrete spectrum. Hence, if s is trace-free, we have $s \perp \{\mathbb{C}\cdot\mathrm{Id}\}$ which implies that

$$\int_X \langle \Delta_{D''}s, s\rangle \geqslant \lambda_1 \int_X |s|^2$$

for λ_1 the first eigenvalue of $\Delta_{D''}$. The result follows from both equations above. □

Before we proceed to the next result, we remember that a $k-$form $\alpha \in \Omega^k(X)$ is called *primitive* if $\Lambda\alpha = 0$. Analogously, we call a bundle valued $k-$form $s \in \Omega^k(E)$ primitive if $\Lambda s = 0$. Next is a Higgs bundle version of the Bogomolov inequality first proved in (SIMPSON, 1988, Proposition 3.4).

Lemma 4. *Let $(\mathcal{E},\theta,H)$ be a Hermitian Higgs bundle over a compact Kähler manifold (X,ω), if denote by $\tilde{F}_H^{\perp}$ the primitive part of $\tilde{F}_H$ then there is a constant $C > 0$ such that*

$$\left(2c_2(H) - \frac{r-1}{r}c_1(H)^2\right)\wedge\omega^{n-2} = C\big(|\tilde{F}_H^{\perp}| - |\tilde{K}_H|\big)\,\mathrm{vol},$$

where $r = \operatorname{rk}\mathcal{E}$ and $c_i(H)$ are the Chern forms associated to the Hitchi-Simpson connection D_H.

Finally, we have one of the main theorems on Higgs bundles, which was proven by Simpson in (SIMPSON, 1988), extending the Hitchin-Kobayashi correspondence to this setting.

Theorem 6 (Simpson)**.** *Let $(\mathcal{E},\theta)$ be a Higgs bundle over a compact Kähler manifold (X,ω). If $(\mathcal{E},\theta)$ is stable then it admits a HHE metric.*

2.3 Formulas for the mean curvature and its derivative

Let $(\mathcal{E}, \theta)$ be a Higgs bundle, H_0 a fixed Hermitian metric, $\nabla_0 = \nabla_{H_0}$ its associated Chern connection and $D_0 = \nabla_0 + \theta + \theta^*$ the connection altered by the Higgs field. In what follow we will compute how the mean curvature behave under a change of metric, these results are based on (LüBKE; TELEMAN, 2006, Section 6.1) and (JACOB; WALPUSKI, 2018, Appendix A).

If $s \in \Gamma(i\mathfrak{su}(E, H_0))$ then $H := H_0 \,\mathrm{e}^s$ defines a Hermitian metric on E and an isometry $\mathrm{e}^{s/2} : (E, H) \to (E, H_0)$. If ∇_s and D_s are the associated connections on (E, H) then we will denote by $\tilde{\nabla}_s$ and $\tilde{D}_s$ their push-forward by $\mathrm{e}^{s/2}$. Set

$$\mathfrak{K}(s) := \mathrm{Ad}(\mathrm{e}^{s/2})\tilde{K}_{H_0 \mathrm{e}^s} = i\Lambda F_{\tilde{D}_s},$$

where $\mathrm{Ad}(\mathrm{e}^s)$ denotes the Lie group action given by conjugation with e^s.

Proposition 6. *The following formula holds*

$$\begin{aligned}\mathfrak{K}(s) =& (2 - \cosh(\mathrm{ad}_{s/2}))\tilde{K}_{H_0} + \frac{1}{2}\Theta(s)\Delta_{D_0} s \\ &+ \frac{i}{2}\Lambda(D_0\Upsilon(-s/2) \wedge D_0' s) - \frac{i}{2}\Lambda(D_0\Upsilon(s/2) \wedge D_0'' s) \\ &- \frac{i}{4}\Lambda(\Upsilon(-s/2)D_0' s \wedge \Upsilon(s/2)D_0'' s + \Upsilon(s/2)D_0'' s \wedge \Upsilon(-s/2)D_0' s)\end{aligned}$$

where ad_s *denotes the Lie algebra action of* s *given by commuting with* s *and* $\Upsilon(s), \Theta(s) \in \mathrm{End}(\mathfrak{gl}(E))$ *are defined by*

$$\Upsilon(s) := \frac{\mathrm{e}^{\mathrm{ad}_s} - \mathrm{Id}}{\mathrm{ad}_s} \quad \text{and} \quad \Theta(s) := \frac{\Upsilon(s/2) + \Upsilon(-s/2)}{2}. \tag{2.12}$$

Proof. Since $D_s' = D_0' + \mathrm{e}^{-s} D_0'(\mathrm{e}^s)$ and $D_s'' = \bar{\partial} + \theta$, we have

$$\begin{aligned}\tilde{D}_s' &= \mathrm{e}^{s/2}(D_0' + \mathrm{e}^{-s} D_0'(\mathrm{e}^s))\,\mathrm{e}^{-s/2} \\ &= D_0' + \mathrm{e}^{s/2} D_0'(\mathrm{e}^{-s/2}) + \mathrm{e}^{-s/2} D_0'(\mathrm{e}^s)\,\mathrm{e}^{-s/2} \\ &= D_0' + \mathrm{e}^{-s/2} D_0'(\mathrm{e}^{s/2})\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\tilde{D}_s'' &= \mathrm{e}^{s/2}(\bar{\partial} + \theta)\,\mathrm{e}^{-s/2} \\ &= \bar{\partial} + \mathrm{e}^{s/2}\,\bar{\partial}(\mathrm{e}^{-s/2}) + \mathrm{e}^{s/2}\,\theta\,\mathrm{e}^{-s/2} \\ &= D_0'' - D_0''(\mathrm{e}^{s/2})\,\mathrm{e}^{-s/2}.\end{aligned} \tag{2.14}$$

Using

$$d_x \exp(y) = (\Upsilon(x)y)\,\mathrm{e}^x = \mathrm{e}^x(\Upsilon(-x)y), \tag{2.15}$$

it follows that

$$\tilde{D}_s = D_0 + \frac{1}{2}\Upsilon(-s/2)D's - \frac{1}{2}\Upsilon(s/2)D''s.$$

Thus we obtain

$$\begin{aligned}F_{\tilde{D}_s} &= F_{D_0} + \frac{1}{2}\Upsilon(-s/2)D''D_0's - \frac{1}{2}\Upsilon(s/2)D_0'D''s \\ &+ \frac{1}{2}D_0\Upsilon(-s/2) \wedge D_0's - \frac{1}{2}D_0\Upsilon(s/2) \wedge D''s \\ &- \frac{1}{4}(\Upsilon(-s/2)D_0's \wedge \Upsilon(s/2)D''s + \Upsilon(s/2)D''s \wedge \Upsilon(-s/2)D_0's).\end{aligned}$$

Applying $i\Lambda$ and using the first two items of Equation 2.8, it follows that

$$\begin{aligned}\mathrm{Ad}(\mathrm{e}^{s/2})\tilde{K}_{H_0\,\mathrm{e}^s} &= \tilde{K}_{H_0} + \frac{1}{4}(\Upsilon(-s/2) - \Upsilon(s/2))[K_{H_0}, s] \\ &+ \frac{1}{4}(\Upsilon(s/2) + \Upsilon(-s/2))\Delta_{D_0}s \\ &+ \frac{i}{2}\Lambda(D_0\Upsilon(-s/2) \wedge D_0's) - \frac{i}{2}\Lambda(D_0\Upsilon(s/2) \wedge D''s) \\ &- \frac{i}{4}\Lambda(\Upsilon(-s/2)D_0's \wedge \Upsilon(s/2)D''s + \Upsilon(s/2)D''s \wedge \Upsilon(-s/2)D_0's).\end{aligned}$$

This implies the asserted identity since

$$1 - \frac{x}{4}\left(\frac{\mathrm{e}^{-x/2}-1}{x/2} + \frac{\mathrm{e}^{x/2}-1}{x/2}\right) = 2 - \cosh(x/2)$$

□

Proposition 7. *The following formula holds*

$$d_s\mathfrak{K}(\hat{s}) = \frac{1}{4}\Delta_{\tilde{D}_s}(\mathrm{Id} + \mathrm{Ad}(\mathrm{e}^{-s/2}))\Upsilon(s/2)\hat{s} - \frac{1}{4}[\mathfrak{K}, (\mathrm{Id} - \mathrm{Ad}(\mathrm{e}^{-s/2}))\Upsilon(s/2)\hat{s}]$$

Proof. We have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} F_{\tilde{D}_{s+t\hat{s}}} = \tilde{D}_s\left(\left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \tilde{D}_s\right)$$

Using (2.13) and (2.15), we compute

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \tilde{D}'_{s+t\hat{s}} &= \frac{1}{2}(\mathrm{e}^{-s/2}\, D_0'(\mathrm{e}^{s/2}\, \mathrm{Ad}(\mathrm{e}^{-s/2})\Upsilon(s/2)\hat{s}) - (\mathrm{Ad}(\mathrm{e}^{-s/2})\Upsilon(s/2)\hat{s})\, \mathrm{e}^{-s/2}\, D_0'\, \mathrm{e}^{s/2}) \\ &= \frac{1}{2}(D_0'(\mathrm{Ad}(\mathrm{e}^{-s/2})\Upsilon(s/2)\hat{s}) + [\mathrm{e}^{-s/2}\, D_0'\, \mathrm{e}^{s/2}, \mathrm{Ad}(\mathrm{e}-s/2)\Upsilon(s/2)\hat{s}]) \\ &= \frac{1}{2}\tilde{D}_s'\, \mathrm{Ad}(\mathrm{e}^{-s/2})\Upsilon(s/2)\hat{s};\end{aligned}$$

and using (2.14) and (2.15), we compute

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \tilde{D}''_{s+t\hat{s}} &= -\frac{1}{2}(D''((\Upsilon(s/2)\hat{s})\, \mathrm{e}^{s/2}) - (D''\, \mathrm{e}^{s/2})\, \mathrm{e}^{-s/2}(\Upsilon(s/2)\hat{s})) \\ &= -\frac{1}{2}(D''(\Upsilon(s/2)\hat{s}) - [(D''\, \mathrm{e}^{s/2})\, \mathrm{e}^{-s/2}, \Upsilon(s/2)\hat{s}]) \\ &= -\frac{1}{2}\tilde{D}_s''(\Upsilon(s/2)\hat{s}).\end{aligned}$$

Thus we obtain

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0}\mathfrak{K}(s+t\hat{s}) &= \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} i\Lambda F_{\tilde{D}_{s+t\hat{s}}}\\ &= \frac{i}{2}\Lambda(\tilde{D}''_s\tilde{D}'_s\,\mathrm{Ad}(\mathrm{e}^{-s/2})\Upsilon(s/2)\hat{s} - \tilde{D}'_s\tilde{D}''_s\Upsilon(s/2)\hat{s})\\ &= \frac{1}{4}i\Lambda(\tilde{D}''_s\tilde{D}'_s - \tilde{D}'_s\tilde{D}''_s)(\mathrm{Id}+\mathrm{Ad}(\mathrm{e}^{-s/2}))\Upsilon(s/2)\hat{s}\\ &- \frac{1}{4}i\Lambda(\tilde{D}''_s\tilde{D}'_s + \tilde{D}'_s\tilde{D}''_s)(\mathrm{Id}-\mathrm{Ad}(\mathrm{e}^{-s/2}))\Upsilon(s/2)\hat{s}\\ &= \frac{1}{4}\Delta_{\tilde{D}_s}(\mathrm{Id}+\mathrm{Ad}(\mathrm{e}\,-s/2))\Upsilon(s/2)\hat{s}\\ &- \frac{1}{4}[i\Lambda F_{\tilde{D}_s}, (\mathrm{Id}-\mathrm{Ad}(\mathrm{e}^{-s/2}))\Upsilon(s/2)\hat{s}]\end{aligned}$$

□

Proposition 8. *The following formula holds*

$$\langle \mathfrak{K}(s) - K_{H_0,\theta}, s\rangle = \langle i\Lambda D''(\mathrm{e}^{-s}\,D'_0\,\mathrm{e}^{s}), s\rangle = \frac{1}{4}\Delta|s|^2 + \frac{1}{2}|\upsilon(-s)D_0 s|^2$$

where $\upsilon(s) \in \mathrm{End}(\mathfrak{gl}(E))$ *is defined by*

$$\upsilon(s) := \sqrt{\Upsilon(s)}. \tag{2.16}$$

Proof. This is a computation

$$\begin{aligned}\langle i\Lambda D''(\mathrm{e}^{-s}\,D'_0\,\mathrm{e}^{s}), s\rangle &= \langle i\Lambda D''(\Upsilon(-s)D'_0 s), s\rangle\\ &= i\Lambda\bar{\partial}\langle\Upsilon(-s)D'_0 s, s\rangle + i\Lambda\langle\Upsilon(-s)D'_0 s, \partial_0 s\rangle + i\Lambda\langle[\theta, \Upsilon(-s)D'_0 s], s\rangle\\ &= i\Lambda\bar{\partial}\langle\partial_0 s, s\rangle + i\Lambda\bar{\partial}\langle\Upsilon(-s)[\theta^{*_0}, s], s\rangle\\ &+ i\Lambda\langle\Upsilon(-s)D'_0 s, \partial_0 s\rangle + i\Lambda\langle[\theta, \Upsilon(-s)D'_0 s], s\rangle\\ &= \frac{1}{4}\Delta|s|^2 + i\Lambda\bar{\partial}\langle[\theta^{*_0}, s], s\rangle + i\Lambda\langle\Upsilon(-s)D'_0 s, D'_0 s\rangle\\ &= \frac{1}{4}\Delta|s|^2 + \frac{1}{2}|\upsilon(-s)D_0 s|^2.\end{aligned}$$

□

2.4 The Donaldson's functional for Higgs bundles

Here we will consider the Donaldson functional on Higgs bundles. After defining the functional, we will present some of its main properties that will be useful further in the text.

Let (X, g, I) be a compact Kähler manifold and $(\mathcal{E}, \theta)$ a Higgs bundle over X. Given a pair of metrics (H, K) we define the Donaldson functional as

$$\mathcal{M}(H, K) := \int_0^1 \int_X \langle s, \mathrm{Ad}(\mathrm{e}^{\frac{us}{2}})\tilde{K}_{H\,\mathrm{e}^{us}}\rangle du,$$

where $s = \log(H^{-1}K)$. This functional first appeared in (DONALDSON, 1985) where it was used to prove the existence of HE metrics for algebraic surfaces. It plays an important role in Donaldson's method since its minimal points correspond to HHE metrics and the gradient flow associated to it is given by Donaldson's heat flow. Although we don't use Donaldson's method in this text, some of the functional's properties will be useful when computing the a priori estimates.

This first result, whose proof can be found in (SIMPSON, 1988, Proposition 5.1), shows an additive property for $\mathcal{M}$.

Lemma 5. *If H_1, H_2, H_3 are Hermitian metrics we have*

$$\mathcal{M}(H_1, H_3) = \mathcal{M}(H_1, H_2) + \mathcal{M}(H_2, H_3).$$

Next we have an upper bound for the functional whose proof can be seen in (DONALDSON, 1987, Lemma 24).

Lemma 6. *There is a constant $C > 0$ such that for any Hermitian metric H and $s \in C^\infty(X, i\mathfrak{su}(E, H))$ we have*

$$\mathcal{M}(H, H\,\mathrm{e}^s) \leqslant C \int_X |s||K_{H\,\mathrm{e}^s}|.$$

Lastly, we have this lower bound for the functional whose proof can be found in (SIMPSON, 1988, Proposition 5.3).

Lemma 7. *If H is a HHE metric then*

$$\|s\|_{L^2} - 1 \leqslant C\mathcal{M}(H, H\,\mathrm{e}^s),$$

where $C > 0$ is a constant that doesn't depend on the metric H.

2.5 Higgs bundles over K3 surfaces

Motivated by the last example of section 1.3, in this section we will discuss examples of Higgs bundles over K3 surfaces. With this we aim to reproduce the same construction in the case of $Spin(7)-$instantons.

As we have seen, we have an ACyl Calabi-Yau $4-$fold $W \times T^2$ with cross-section $X \times T^2$, for a K3 surface X, and over $W \times T^2$ we have an ATI holomorphic Higgs bundle $(\mathcal{E}, \theta)$ (trivial along T^2) asymptotic to a Higgs bundle $(\mathcal{E}_{X\times T^2}, \theta_{X\times T^2})$ (also trivial along T^2). In this setting, the existence of an HHE metric is assured if one has stability for the bundle $(\mathcal{E}_{X\times T^2}, \theta_{X\times T^2})$. Hence to obtain examples of $\mathrm{Spin}(7)-$instantons, we need to know examples of stable Higgs bundles over $X \times T^2$.

The following theorem, due to Biswas et al., characterizes all HHE metrics in the context of Calabi-Yau manifolds and helps understanding possible examples.

Theorem 7 ((Biswas et al., 2016, Theorem 3.3))**.** *Let $(\mathcal{E}, \theta)$ be a polystable Higgs bundle over a compact Calabi-Yau manifold X and suppose that H is a Higgs-Hermitian-Einstein metric. Then H is a Hermite-Einstein metric for $\mathcal{E}$.*

In particular, we conclude from the theorem above that every holomorphic bundle $\mathcal{E}$ underlying a polystable Higgs bundle $(\mathcal{E}, \theta)$ is also polystable.

Returning to our case, since the Higgs bundle $(\mathcal{E}_{X\times T^2}, \theta_{X\times T^2})$ is T^2 invariant, the Higgs field has the form

$$\theta_{X\times T^2} = T dz, \tag{2.17}$$

where $T \in \mathrm{End}(\mathcal{E})$. Moreover, since $W \times T^2$ is Calabi-Yau, it follows from Theorem 7 that the vector bundle $\mathcal{E}_{X\times T^2}$ is polystable. Hence we have

$$\mathcal{E}_{X\times T^2} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k,$$

where each $\mathcal{E}_i$ is a stable holomorphic vector bundle, all with the same slope.

Decomposing the endomorphism T in components $T_{ij} : \mathcal{E}_i \to \mathcal{E}_j$ we obtain a collection of maps between stable vector bundles of same slope. It follows from Proposition 4 that each T_{ij} is either zero or an isomorphism, hence either every T_{ij} is zero or there is i, j such that T_{ij} is an isomorphism. It follows that we can identify the isomorphic summands and write without loss of generality

$$\mathcal{E}_{X\times T^2} = \mathcal{E}_1^{\oplus m_1} \oplus \cdots \oplus \mathcal{E}_l^{\oplus m_l},$$

where each $\mathcal{E}_i$ is stable and not isomorphic to $\mathcal{E}_j$ for $i \neq j$. Thus we can see T as a block diagonal matrix

$$(T) = \begin{pmatrix} (T_1) & 0 & \cdots & 0 \\ 0 & (T_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (T_l) \end{pmatrix}$$

where each (T_i) is a $m_i \times m_i$ matrix. This means that each pair $(\mathcal{E}_i^{\oplus m_i}, T_i dz)$ is a Higgs subbundle of $(\mathcal{E}_{X\times T^2}, \theta_{X\times T^2})$. Thus, since they have the same slope, by stability we must have $(\mathcal{E}_{X\times T^2}, \theta_{X\times T^2}) = (\mathcal{E}_1^{\oplus m_1}, T_1 dz)$.

Recalling that a stable Higgs bundle is also simple as a Higgs bundle (see Corollary 3) and using the fact that T_1 commutes with $T_1 dz$, we see that $T_1 = \lambda \cdot \mathrm{Id}$. Therefore we have the following result:

Proposition 9. *If $(\mathcal{E}_{X\times T^2}, \theta_{X\times T^2})$ is a T^2 invariant stable Higgs bundle then $\theta_{X\times T^2} = \lambda \cdot \mathrm{Id}\, dz$.*

3 ACyl geometry

In this chapter I will provide the basic background about ACyl Kähler manifolds, ATI bundles and ATI Hermitian metrics. The goal is to introduce the main definitions and results used in the solution of the HHE problem. Our main references for this chapter are (HASKINS; HEIN; NORDSTRöM, 2015), (CORTI et al., 2013) and (CORTI et al., 2015).

3.1 ACyl Kähler manifolds

In this section we introduce some notations and the basic definitions related to ACyl manifolds. The content present here is mostly based on (JACOB; WALPUSKI, 2018; EARP, 2018).

Before we define ACyl manifolds, let us remind that we call a *cylinder* any manifold of form $(a,b)\times X$, where (a,b) is a connected open subset of $\mathbb{R}$ and X is a closed manifold. Moreover, we denote by l the coordinate function given by the projection $p_1 : (a,b)\times X \to (a,b)$ and by $p_2 : (a,b)\times X \to X$ the projection on X. For a vector bundle E_X over X, we will often denote by $E_\times$ its pullback by p_2 and for a section $s_X \in \Gamma(E_X)$ we will denote by $s_\times$ the corresponding pullback.

Due to their geometric simplicity, cylinders have at their disposal various tools to treat analytical problems, unlike most of the other non-compact manifolds. Based on this, the following definition tries to extend the use of such tools to a more general setting.

Definition 13. *A manifold W is said to have* tubular end *if there is a diffeomorphism $\pi : W\backslash K \to \mathbb{R}_{>1}\times X$, where K is a compact subset of W and $\mathbb{R}_{>1}\times X$ a cylinder. In this case, we call $W\backslash K$ the* cylindrical end *of W, π the* tubular model *and X the* asymptotic cross-section*.*

If W have tubular end, we can smoothly extend $l\circ\pi : W\backslash K \to \mathbb{R}_{>1}$ to a function, which we will denote by $l : W \to \mathbb{R}_{\geqslant 0}$, such that $l \leqslant 1$ on K. For $L \geqslant 1$, we define the truncated manifold

$$W_L := l^{-1}([0,L]).$$

Since a cylinder may have many connected components, one for each components of X, a manifold with tubular end may have many topological ends. The following theorem shows that the situation is much more simple in the setting of Ricci-flat manifolds.

Theorem 8 ((SALUR, 2006, Theorem 1))**.** *If a connected and orientable manifold W have tubular end and admits a Ricci-flat metric, then the number of topological ends of*

W is less than or equal to 2. Moreover, the number is equal to 2 if and only if W is a cylinder.

Thus, since we are mostly interested in manifolds with special holonomy which are not cylinders, we will assume henceforth that our cross-section X is connected so that W has only one topological end. Now that this topological inconvenience has been settled, we may introduce the notion of asymptotic behaviour to the tubular setting.

Definition 14. *Let E be a vector bundle over a manifold with tubular end W and E_X a vector bundle over X. We say that E is* asymptotic *to E_X if there is a bundle isomorphism $\bar{\pi} : E|_{W\backslash K} \to E_\times$ covering the tubular model π. We call a vector bundle $E \to W$* asymptotically translation-invariant *if it is asymptotic to some vector bundle over X.*

Now, suppose that $E \to W$ is asymptotic to E_X and let $H_\times$ be the pullback of a metric H_X on E_X. We say that $s \in \Gamma(E)$ is *asymptotic* to $s_X \in \Gamma(E_X)$ *at rate* $\delta > 0$ ($s \overset{\delta}{\rightsquigarrow} s_X$) if for all $k \in \mathbb{N}$ there is $C_k > 0$ such that

$$|\nabla^k_{H_\times}(\pi_* s - s_\times)|_{H_\times} \leqslant C_k \,\mathrm{e}^{-\delta l}\,.$$

Note that, since X is compact, the definition doesn't depend on the choice of H_X. We say that s is *aymptotic* to s_X ($s \rightsquigarrow s_X$) if $s \overset{\delta}{\rightsquigarrow} s_X$ for some $\delta > 0$. Finally, we call $s \in \Gamma(E)$ *asymptotically translation-invariant* (ATI) if $s \rightsquigarrow s_X$ for some $s_X \in \Gamma(E_X)$.

For a manifold W with tubular end, the diffeomorphism $\pi : W\backslash K \to \mathbb{R}_{>1} \times X$ clearly defines a bundle isomorphism $\bar{\pi} : T(W\backslash K) \to T\mathbb{R}_{>1} \times TX$. Hence, using that

$$T\mathbb{R}_{>1} \times TX \simeq p_2^*(X \times \mathbb{R} \oplus TX),$$

it follows that TW is an asymptotically translation-invariant bundle asymptotic to $X \times \mathbb{R} \oplus TX$. Moreover, since bundle operations preserves this construction, this also implies that the tensor bundle $T^{r,s}W$ is asymptotic to $\otimes_r(X \times \mathbb{R} \oplus TX) \otimes_s (X \times \mathbb{R} \oplus TX)^*$. With this in mind, we will always use this setting when dealing with ATI tensors over W.

Following the notation described above, we can finally define ACyl Kähler manifolds.

Definition 15. *A Kähler manifold (W, g, I) is called* asymptotically cylindrical *(ACyl) if W has a tubular end diffeomorphic to $\mathbb{R}_{>1} \times S^1 \times X$ for some compact Kähler manifold (X, g_X, I_X) such that*

$$g \rightsquigarrow \mathrm{d}\,l^2 \oplus \mathrm{d}\,s^2 \oplus g_X \quad \text{and} \quad I \rightsquigarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus I_X,$$

where (l, s) are the canonical coordinates on $(0, \infty) \times S^1$ which we shall identify with $\mathbb{C}^*$. The greatest $\delta > 0$ such that the above decays hold is called the decaying rate *of (W, g, I) and is denoted by δ_W.*

Identifying $(1,\infty)\times S^1$ with a subset of $\mathbb{C}^*$, we define

$$X_z := \pi^{-1}(\{z\}\times X),$$

for all $|z|>1$.

Now that we have the notion of a ACyl Kähler manifold, we can extend the definition of ATI bundle to Higgs bundles. For this recall that for a Higgs bundle $(\mathcal{E},\theta)$ we denote by E the underlying smooth vector bundle and by $\bar\partial$ the complex structure.

Definition 16. *A Higgs bundle $(\mathcal{E},\theta)\to W$ is called* asymptotically translation-invariant *(ATI) if $E\to W$ is asymptotic to $E_X\to X$ for some Higgs bundle $(\mathcal{E}_X,\theta_X)$ such that*

$$\bar\partial \rightsquigarrow \bar\partial_X \quad \text{and} \quad \theta \rightsquigarrow \theta_X.$$

In this case we also say that $(\mathcal{E},\theta)\to W$ is asymptotic to *$(\mathcal{E}_X,\theta_X)$. The greatest $\delta>0$ such that the above decays hold is called the* decaying rate *of $(\mathcal{E},\theta)$ and is denoted by $\delta_{(\mathcal{E},\theta)}$.*

Finally, we define asymptotically translation-invariant metrics.

Definition 17. *A Hermitian metric H on a ATI vector bundle $E\to W$ asymptotic to $E_X\to X$ is called* asymptotically translation-invariant *(ATI) if*

$$H\rightsquigarrow H_X$$

for some Hermitian metric H_X on E_X.

3.2 Functional analysis on ACyl manifolds

In this section we introduce all the analytical tools related to ACyl manifolds that will be used posteriorly. The primary references for the material in this section are (LOCKHART; MCOWEN, 1985; HASKINS; HEIN; NORDSTRöM, 2015; PACINI, 2013).

Let's start fixing an ATI Higgs bundle $(\mathcal{E},\theta)\to(W,g,I)$ asymptotic to $(\mathcal{E}_X,\theta_X)\to(X,g_X,I_X)$ and an ATI Hermitian metric H asymptotic to H_X. For each $k\in\mathbb{N}$ we define $C^k(E)$ to be the space of continuous sections s on E having k continuous derivatives and satisfying

$$\|s\|_{C^k}=\sum_{i=0}^{k}\sup_{x\in W}|\nabla^i s|<\infty,$$

where ∇^i denotes the tensor powers of the Chern connection induced by H and the Levi-Civita connection induced by g.

Now, since (W,g,I) has bounded geometry, its injectivity radius $\iota(g)$ is positive. If we denote by $P_{x,y}$ the parallel transport of ∇ along the only geodesic joining x to y (

for $d(x,y) < \iota(g)$), we set, for $0 < \alpha \leqslant 1$,

$$[s]_\alpha = \sup_{\substack{x \neq y \in W \\ d(x,y) < \iota(g)}} \frac{|s(x) - P^*_{x,y} s(y)|}{d(x,y)^\alpha}.$$

Hence we define, for $k \in \mathbb{N}$ and $0 < \alpha \leqslant 1$, the *Hölder space*

$$C^{k,\alpha}(E) := \left\{ s \in C^k(E) : \|s\|_{C^{k,\alpha}} < \infty \right\},$$

where

$$\|s\|_{C^{k,\alpha}} = \sum_{i=0}^{k} \sup_{x \in W} |\nabla^i s| + [\nabla^k s]_\alpha.$$

Although the Hölder spaces are the classical Banach spaces to study regular solutions of PDE's, in the context of ACyl manifolds we need a more restricted class of spaces. Hence we consider the following function spaces.

Definition 18. *We define, for $k \in \mathbb{N}$, $0 < \alpha < 1$ and $\delta > 0$, the* weighted Hölder space

$$C^{k,\alpha}_\delta(E) := \left\{ s \in C^{k,\alpha}(E) : \|s\|_{C^{k,\alpha}_\delta} < \infty \right\}, \tag{3.1}$$

where

$$\|s\|_{C^{k,\alpha}_\delta} := \|\mathrm{e}^{\delta l} \cdot s\|_{C^{k,\alpha}},$$

and denote

$$C^\infty_\delta(E) := \bigcap_{k \in \mathbb{N}} C^{k,\alpha}_\delta(E). \tag{3.2}$$

Denoting by $i\mathfrak{su}(E,H)$ the bundle of self-adjoint endomorphisms of (E,H), we define $C^{k,\alpha}_\delta(i\mathfrak{su}(E,H))$ and $C^\infty_\delta(i\mathfrak{su}(E,H))$ in a similar manner.

One can define another norm with exponential factor on $C^{k,\alpha}(E)$ given by

$$\|s\|_{C^{k,\alpha}_{\delta,*}} := \sum_{i=0}^{k} \sup_{x \in W} |\mathrm{e}^{\delta l} \nabla^i s| + [\mathrm{e}^{\delta l} \nabla^k s]_\alpha.$$

Although it simpler to compute since it doesn't involve derivatives of $\mathrm{e}^{\delta l}$, the following result shows that these are in fact equivalent.

Lemma 8. *The norms $\|\cdot\|_{C^{k,\alpha}_\delta}$ and $\|\cdot\|_{C^{k,\alpha}_{\delta,*}}$ are equivalent.*

Proof. First, we need to estimate $\mathrm{e}^{\delta l}$ and its derivatives. Observe that

$$\nabla^k(\mathrm{e}^{\delta l}) = \mathrm{e}^{\delta l} P_k(\delta \nabla l, \ldots, \delta \nabla^k l),$$

where $P_n(x_1, \ldots, x_n)$ is a polynomial (with positive coefficients) satisfying the recursion

$$P_0 = 1 \quad \text{and} \quad P_n(x_1, \ldots, x_n) = x_1 P_{n-1}(x_1, \ldots, x_{n-1}) + \sum_{k=1}^{n-1} \partial_k P_{n-1}(x_1, \ldots, x_{n-1}) x_{k+1}.$$

Since in the tubular model the function l is the projection, which grows linearly along the tube, we have $\|l\|_{C^{k,\alpha}} < \infty$ for all $k \in \mathbb{N}, 0 < \alpha \leqslant 1$. Hence,

$$\begin{aligned}|\nabla^k(\mathrm{e}^{\delta l})| &\leqslant \mathrm{e}^{\delta l}\, P_k(\delta|\nabla l|, \ldots, \delta|\nabla^k l|)\\ &\leqslant c_k\, \mathrm{e}^{\delta l},\end{aligned}$$

for $c_k = P_k(\delta\|l\|_{C^k}, \ldots, \delta\|l\|_{C^k})$.

Now, if $s \in \Gamma(E)$, we have, by Lemma 16,

$$\begin{aligned}|\nabla^n(\mathrm{e}^{\delta l}\, s)| &\leqslant \sum_{k=0}^{n} \binom{n}{k} |\nabla^{n-k}(\mathrm{e}^{\delta l})||\nabla^k s|\\ &\leqslant \sum_{k=0}^{n} \binom{n}{k} c_{n-k}\, \mathrm{e}^{\delta l}|\nabla^k s|\\ &\leqslant \tilde{c}_n \sum_{k=0}^{n} |\mathrm{e}^{\delta l}\, \nabla^k s|,\end{aligned}$$

for $\tilde{c}_n = \max\limits_{k=1,\ldots,n} \binom{n}{k} c_{n-k}$ which implies

$$\|s\|_{C^{m,\alpha}_\delta} \leqslant \sum_{n=0}^{m} \tilde{c}_n \|s\|_{C^{n,\alpha}_{\delta,*}} \leqslant \left(\sum_{n=0}^{m} \tilde{c}_n\right) \|s\|_{C^{m,\alpha}_{\delta,*}}.$$

On the other hand, notice that

$$|\mathrm{e}^{\delta l}\, \nabla s| \leqslant |\nabla(\mathrm{e}^{\delta l}\, s)| + |\delta\, \mathrm{e}^{\delta l}(\nabla l)s| \leqslant |\nabla(\mathrm{e}^{\delta l}\, s)| + c_1|\mathrm{e}^{\delta l}\, s|,$$

implies

$$\|\mathrm{e}^{\delta l}\, \nabla s\|_{C^0} \leqslant (c_1 + 1)\|s\|_{C^{1,\alpha}_\delta}.$$

Thus, using Lemma 16 and induction, we conclude that

$$\begin{aligned}\|\mathrm{e}^{\delta l}\, \nabla^n s\|_{C^0} &\leqslant \|\nabla^n(\mathrm{e}^{\delta l}\, s)\|_{C^0} + \sum_{k=0}^{n-1} \|\nabla^{n-k}(\mathrm{e}^{\delta l})\nabla^k s\|_{C^0}\\ &\leqslant \|\nabla^n(\mathrm{e}^{\delta l}\, s)\|_{C^0} + \left(\sum_{k=0}^{n-1}(c_{n-k} + 1)\right)^n \|s\|_{C^{n-1,\alpha}_\delta}\\ &\leqslant \left(\sum_{k=0}^{n-1}(c_{n-k} + 1)\right)^n \|s\|_{C^{n,\alpha}_\delta}\end{aligned}$$

obtaining

$$\|s\|_{C^{n,\alpha}_{\delta,*}} \leqslant n\left(\sum_{k=0}^{n-1}(c_{n-k} + 1)\right)^n \|s\|_{C^{n,\alpha}_\delta}.$$

□

We start by adapting Arzelà-Ascoli theorem to the context of ACyl manifolds. For this, let $E \to W$ be an ATI bundle

Lemma 9. *For all $k \in \mathbb{N}$ the inclusion*

$$C^{k+1}_{\delta}(E) \hookrightarrow C^k(E)$$

is compact.

Proof. To show that the inclusion is compact we will show that the ball

$$B_r = \{s \in C^{k+1}_{\delta}(E) : \|s\|_{C^{k+1}_{\delta}} \leqslant r\}$$

is sequentially compact in the C^k norm. So let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence contained in B_r and fix $L \geqslant 1$. Since W_L is compact and $\|s_n\|_{C^k} \leqslant \|s_n\|_{C^{k+1}_{\delta}} \leqslant r$, it follows from Theorem 18 that $\{s_n\}_{n\in\mathbb{N}}$ admits a subsequence $\{s_{n_0}\}_{n_0\in\mathbb{N}}$ that converges in C^k over W_L. Repeating this process inductively, we obtain a chain of sequences $\{s_{n_0}\}_{n_0\in\mathbb{N}} \supset \cdots \supset \{s_{n_i}\}_{n_i\in\mathbb{N}} \supset \dots$ such that $\{s_{n_i}\}_{n_i\in\mathbb{N}}$ converges in C^k to some $\tilde{s}_i$ over W_{L+i}. By uniqueness of the limit, we have $\tilde{s}_i = \tilde{s}_0$, further $|\tilde{s}_0| \leqslant r\,\mathrm{e}^{l}$ by point-wise convergence.

Now, let $\{s_m\}_{m\in\mathbb{N}}$ be the diagonal sequence whose mth term is the mth term of the mth subsequence. Given $\epsilon > 0$ we have

$$|\tilde{s}_0 - s_m| \leqslant |\tilde{s}_0| + |s_m| \leqslant 2r\,\mathrm{e}^{-\delta l} < \frac{\epsilon}{2}$$

for $l > -\delta^{-1}\ln(\epsilon/4r)$. Moreover, there is $M_\epsilon \in \mathbb{N}$ such that

$$\|\tilde{s}_0 - s_m\|_{C^k} < \frac{\epsilon}{2} \quad \forall m \geqslant M_\epsilon$$

over $W_{-\delta^{-1}\ln(\epsilon/4r)}$. Thus,

$$\|\tilde{s}_0 - s_m\|_{C^k} < \epsilon \quad \forall m \geqslant M_\epsilon$$

which implies that $\{s_m\}_{m\in\mathbb{N}}$ converges in C^k to $\tilde{s}_0$. ☐

Now, we will present some results on asymptotically translation-invariant operators on ACyl manifolds. This first result is a combination of (HASKINS; HEIN; NORDSTRöM, 2015, Proposition 2.7) and (JACOB; WALPUSKI, 2018, Proposition 2.7).

Proposition 10. *Let (W, g, I) be an ACyl Kähler manifold and take $C = \min(\delta_W, \sqrt{\lambda_1})$, where $\lambda_1 > 0$ is the smallest eigenvalue of the Laplacian acting on functions. Then, for all $0 < \delta < C$, the linear map $\Delta : C^{k+2,\alpha}_{\delta}(W) \oplus \mathbb{R} \to C^{k,\alpha}_{\delta}(W)$ is injective and its image is the subspace of mean value zero functions. Moreover, if we consider the map*

$$(f, A) \mapsto \Delta f - A\Delta l,$$

then we obtain an isomorphism.

Next we have sufficient condition for Fredholm operators. For this we recall that $\delta \in \mathbb{R}$ is called a *critical weight* of an ATI elliptic operator $\Psi : \Gamma(E) \to \Gamma(F)$ if there is $\lambda \in \mathbb{R}$ such that

$$\Psi_\times(\mathrm{e}^{(-\delta+\mathrm{i}\lambda)l} f) = 0$$

admits a non-zero solution. Here $\Psi_\times$ denotes the translation-invariant operator that Ψ is asymptotic to. Now, we can state the result.

Proposition 11 ((HASKINS; HEIN; NORDSTRöM, 2015, Proposition 2.4))**.** *Let $\Psi : \Gamma(E) \to \Gamma(F)$ be an elliptic ATI operator of order r. If δ is not a critical weight of Ψ then the induced linear map $\Psi : C^{k+r,\alpha}_\delta(E) \to C^{k,\alpha}_\delta(F)$ is Fredholm.*

Using a formula for computing the index of ATI operators, we have the following result due to Lockhart and McOwen.

Theorem 9 ((LOCKHART; MCOWEN, 1985, Theorem 7.4))**.** *Let $\Psi : \Gamma(E) \to \Gamma(E)$ be a selfadjoint ATI elliptic operator and take $\epsilon > 0$ such that $\Psi : C^{k+r,\alpha}_\delta(E) \to C^{k,\alpha}_\delta(F)$ is Fredholm for $\epsilon > |\delta| > 0$. Then, if Ψ is Fredholm for $\delta = 0$ we have that its index is 0.*

Finally, we show that for δ small the Laplacian Δ_{D_H} is Fredholm of index 0, adapting (JACOB; WALPUSKI, 2018, Proposition 2.8) for Higgs bundles.

Proposition 12. *Suppose that $(\mathcal{E}_X, \theta_X)$ is a stable Higgs bundle with HHE metric H_X. Then there is $C = C(H_X) > 0$ such that for all $|\delta| < C$ the linear operator Δ_{D_H} : $C^{k+2,\alpha}_\delta(W, \mathrm{i}\,\mathfrak{su}(E,H)) \to C^{k,\alpha}_\delta(W, \mathrm{i}\,\mathfrak{su}(E,H))$ is Fredholm of index zero.*

Proof. We use the same ideas in (JACOB; WALPUSKI, 2018, Proposition 2.8). First, notice that Δ_{D_H} is asymptotic to the translation-invariant linear operator

$$-\partial_l^2 - \partial_s^2 + \Delta_{D_{H_X}}$$

acting on sections of $\mathrm{i}\,\mathfrak{su}(E_\times, H_\times)$. Thus, using that H_X is HHE and (2.8), we have

$$\frac{1}{2}\Delta_{D_{H_X}} = \Delta_{D'_{H_X}} = \Delta_{D''_{H_X}}.$$

Now, since

$$(\mathcal{E}_X, \theta_X)$$

is stable, we know that $(\mathcal{E}_X, \theta_X)$ is simple which is equivalent to $\Delta_{D''_{H_X}}$ being invertible. Hence, using the ellipticity and compactness of $\Delta_{D_{H_X}}$, we see that the spectrum of $-\partial_\theta^2 + \Delta_{D_{H_X}}$ is contained in $[C^2, \infty)$, for some $C = C(H_X) > 0$. This means that each $|\delta| < C$ is not a critical weight, so $\Delta_{D_{H_X}}$ are Fredholm for each $|\delta| < C$ by Proposition 11. Since $\Delta_{D_{H_X}}$ is formally self-adjoint and 0 is not a critical weight, the index is zero by Theorem 9. □

3.3 Manifolds and bundles with bounded geometry

One of the main characteristics of ACyl manifolds is that they have a bounded geometry. This gives them an uniform geometric control which allow us to do analysis without having to worry about local results not extending globally. Most of this section is based on (GROSSE; SCHNEIDER, 2013) and (SCHICK, 2001).

Definition 19. *A Riemannian manifold (X, g) is said to be of* bounded geometry *if the following two conditions are satisfied:*

- *The injectivity radius* $\operatorname{inj}(X)$ *of (X, g) is positive;*
- *For all $k \in \mathbb{N}$ there is $C_k > 0$ such that*

$$|\nabla^k R| \leqslant C_k,$$

where ∇ is the Levi-Civita connection, R the Riemannian curvature and $|\cdot|$ the norm all induced by g.

It follows from the first condition that a manifold with bounded geometry (X, g) is always complete. Moreover, we have for any $R < \operatorname{inj}(X)$ a special atlas formed by geodesic coordinates over balls of radius R.

Definition 20. *A* geodesic atlas *on a Riemannian manifold (X, g) is an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ where each U_α is a geodesic ball of radius r, for a fixed $0 < r < \operatorname{inj}(X)$, and each $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a geodesic coordinate around U_α.*

Every Riemannian manifold with bounded geometry can be covered by geodesics atlas by the first condition. Besides that, using the uniform bounds given by the second condition, we have the following result which gives a coordinate-wise description of bounded geometry.

Theorem 10 ((EICHHORN, 1991, Theorem A))**.** *Let (X, g) be a manifold with bounded geometry and $\mathcal{A}$ a geodesic atlas. Then for each $k \in \mathbb{N}$ there is $C_k > 0$ such that*

$$|D^\gamma \Gamma^l_{ij}| \leqslant C_k,$$

for all multi index $|\gamma| \leqslant k$. Here Γ denotes the Christoffel symbols in normal coordinate.

With this result any analytical estimate can be locally translated from a ball $B_R(0) \subset \mathbb{R}^n$ to a geodesic ball in $B_R(x) \subset X$. Now we shall extend the theorem above for vector bundles, for this we need the notion of bounded geometry for bundles too.

Definition 21. *A Hermitian vector bundle $(\mathcal{E}, H)$ over a complex manifold with bounded geometry (X, g, I) is said to be of* bounded geometry *if for all $k \in \mathbb{N}$ there is $C_k > 0$ such that*

$$|\nabla^k_H F_{\nabla_H}| \leqslant C_k,$$

where ∇^k is the connection induced by g and H, F_{∇_H} the Chern connection curvature and $|\cdot|$ the norm induced by g and H.

As in the case of geodesic atlas, for vector bundles with bounded geometry we also have a standard way to obtain trivializations using a geodesic atlas and the parallel along geodesic rays.

Definition 22. *Let $(\mathcal{E}, H)$ be a Hermitian vector bundle with bounded geometry over (X, g, I) and let $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ be a geodesic atlas. If $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a normal coordinate denote by x_α the center of U_α. For every U_α in the atlas we can define a trivialization $\psi_\alpha : \mathcal{E}|_{U_\alpha} \to U_\alpha \times \mathbb{C}^k$ by taking the parallel transport of vectors in E_{x_α} along radial geodesics using the connection ∇_H. We call this type of trivialization a* synchronous trivialization*, and we say that $\mathcal{A}_{\mathcal{E}} = \{(U_\alpha, \phi_\alpha, \psi_\alpha)\}$ is a* geodesic atlas *of $\mathcal{E}$.*

Having defined the appropriate trivializations, we can now enunciate a theorem similar to Theorem 10 for vector bundles.

Theorem 11 ((EICHHORN, 1991, Theorem B))**.** *Let $(\mathcal{E}, H)$ be Hermitian bundle with bounded geometry over a complex manifold with bounded geometry (X, g) and let $\mathcal{A}_{\mathcal{E}}$ denote a synchronous trivialization associated to a geodesic atlas $\mathcal{A}$. Then for each $k \in \mathbb{N}$ there is $C_k > 0$ such that*

$$|D^\gamma A^l_{ij}| \leqslant C_k,$$

for all multi index $|\gamma| \leqslant k$. Here A denotes the connection matrix in the synchronous trivialization.

It follows from the theorem above that on normal coordinates, we can use the $C^{k,\alpha}$ norm on the ball $B_r(0)$ in the euclidean space to estimate the $C^{k,\alpha}$ norm in E and the other way around, we can use the $C^{k,\alpha}$ norm in E to estimate the corresponding $C^{k,\alpha}$ norm on the ball $B_r(0)$. Hence we have the following corollary.

Corollary 5. *Let $\mathcal{E} \to X$ be a vector bundle with bounded geometry and denote by $\phi_x : B_r(x) \to B_r(0)$ the normal coordinates around $x \in X$ for $r < \mathrm{inj}(X)$. For all $k \in \mathbb{N}$ and $\alpha \in (0, 1]$ we have constants $a_{k,\alpha}, b_{k,\alpha} > 0$ such that*

$$a_{k,\alpha}|s|_{C^{k,\alpha}(B_r(x))} \leqslant |\phi_x^* s|_{C^{k,\alpha}(B_r(0))} \leqslant b_{k,\alpha}|s|_{C^{k,\alpha}(B_r(x))} \tag{3.3}$$

for any $s \in \Gamma(E)$.

Let $L : C^\infty(E) \to C^\infty(E)$ be a linear differential operator of order m on a bundle with bounded geometry $\mathcal{E} \to X$. Using a geodesic atlas $\mathcal{A}_{\mathcal{E}} = \{(U_\alpha, \phi_\alpha, \psi_\alpha)\}$ we

4 Weak Higgs-Hermite-Einstein metrics on ATI bundles over ACyl manifolds

The main purpose of this chapter is to establish the existence of weak HHE metrics for ATI Higgs bundles over ACyl Kähler manifolds. As discussed previously in subsection 1.4.1, HHE metrics play an important role in obtaining examples of $\mathrm{Spin}(7)$ −instantons. And ATI Higgs bundles are a natural setting to study the existence of those metrics, having in mind the previous works (EARP, 2015; JACOB; WALPUSKI, 2018). Thus, assuming a stability condition for the bundle at infinity, we prove the following theorem

Theorem 12. *Let (W, g, I) be an ACyl Kähler manifold with cross-section (X, g_X, I_X) and $(\mathcal{E}, \theta) \to W$ an ATI Higgs bundle asymptotic to a Higgs bundle $(\mathcal{E}_X, \theta_X) \to X$. If $(\mathcal{E}_X, \theta_X)$ is stable then $(\mathcal{E}, \theta)$ admits an asymptotically translation-invariant Higgs-Hermite-Einstein metric.*

To prove this result, we use the continuity method introduced by Uhlenbeck and Yau (UHLENBECK; YAU, 1986) in their proof of the Hitchin-Kobayashi correspondence. More recently, this method was adapted by Jacob and Walpuski in (JACOB; WALPUSKI, 2018) to the context of ATI holomorphic bundles over ACyl Kähler manifolds. Guided by this last work, we extend the method to the setting of ATI Higgs bundles which we describe below. For a systematic and self-contained reference on the Kobayashi-Hitchin correspondence and the continuity method involved, we refer the reader to the book of Lübke and Teleman (LüBKE; TELEMAN, 1995).

We start considering an ATI Higgs bundle $(\mathcal{E}, \theta)$ over an ACyl Kähler manifold (W, ω) which is asymptotic to a stable Higgs bundle $(\mathcal{E}_X, \theta_X) \to (X, \omega_X)$. Using the stability condition, we are able to construct a Hermitian metric H_0 on $\mathcal{E}$ which asymptotic to a HHE metric on $\mathcal{E}_X$. With it we define a PDE $\mathfrak{L}(s, t) = 0$ (thought as a family in $t \in [0, 1]$ of PDEs) for sections $s \in C^\infty_\delta(W, i\mathfrak{su}(E, H_0))$ such that $\mathfrak{L}(s, 0) = 0$ is precisely the condition for $H_0 \, \mathrm{e}^s$ be a weak HHE metric. The proof then follow 3 steps.

First, using a trick discovered by Itoh and Nakajima (ITOH; NAKAJIMA, 1990), we show that $\mathfrak{L}(s, t) = 0$ admits a solution for $t = 1$. Then, linearising the equation and applying the implicit function theorem, we prove that the set of $t \in [0, 1]$ satisfying $\mathfrak{L}(s, t) = 0$ is open. Finally, we compute a priori estimates obtaining uniform $C^{k,\alpha}_\delta$ bounds which guarantees that the set is also closed. Thus, the set is the whole interval and we conclude that the bundle $(\mathcal{E}, \theta)$ admits a weak HHE metric.

4.1 The continuity method

In this section we begin the proof of Theorem 12 introducing the continuity method. After fixing some definitions and presenting our PDE $\mathfrak{L}(s,t) = 0$, we prove that it admits a solution for $t = 1$. Let us start by settling our framework.

Let (W, g, I) be an ACyl Kähler manifold with cross-section (X, g_X, I_X) and $(\mathcal{E}, \theta)$ an ATI Higgs bundle asymptotic to a stable Higgs bundle $(\mathcal{E}_X, \theta_X)$. Fixing a Hermitian metric H_0 in $\mathcal{E}$, let $D_0 = D_0' + D_0'' = \nabla_{H_0} + \theta + \theta^*$ denote the Hitchin-Simpson connection of $(\mathcal{E}, \theta, H_0)$, F_0 the curvature of D_0 and K_0 the mean curvature. For any other metric H on $\mathcal{E}$ there is $s \in C^\infty(W, i\mathfrak{su}(E, H_0))$, the space of H_0-self-adjoint endomorphisms of E, such that $H(u,v) = H_0(\mathrm{e}^s \cdot u, v)$ for all $u, v \in \Gamma(E)$. Moreover, using (2.11), we have the mean curvature associated to H given by

$$K_H = K_0 + i\Lambda_\omega\big(D''(\mathrm{e}^{-s}\, D_0'(\mathrm{e}^s))\big).$$

Since we are looking for H satisfying the weak HHE condition

$$K_H = \frac{\operatorname{tr} K_H}{\operatorname{rk} E} \cdot \mathrm{Id}_E,$$

setting

$$\tilde{K}_H := K_H - \frac{\operatorname{tr} K_H}{\operatorname{rk} E} \cdot \mathrm{Id}_E$$

turns the condition into

$$\tilde{K}_0 + i\Lambda_\omega\big(D''(\mathrm{e}^{-s}\, D_0'(\mathrm{e}^s))\big) = 0.$$

Thus we obtain a PDE on s whose solution is precisely the condition for $H = H_0\,\mathrm{e}^s$ satisfy the weak HHE condition. To solve this equation we use the continuity method which consists in solving the perturbed equation

$$\tilde{K}_0 + i\Lambda_\omega\big(D''(\mathrm{e}^{-s}\, D_0'(\mathrm{e}^s))\big) + t \cdot s = 0, \tag{4.1}$$

for $t \in [0,1]$. Although this may seem counter intuitive, the equation above is easier to solve, and once a solution is obtained, its restriction for $t = 0$ gives our desired result.

For future applications, we will be interested in solutions of (4.1) having an exponential decay. Thus we will need to change the PDE above to accommodate this restriction to the solution space. Denote by $\delta_W, \delta_{\mathcal{E},\theta}$ the decaying rates of (W, g, I), $(\mathcal{E}, \theta)$ (see Definition 15 and Definition 16), by λ_X the first eigenvalue of the Laplacian on (X, g_X, I_X), and fix

$$0 < \delta < \min\{\delta_W, \delta_{\mathcal{E},\theta}, \sqrt{\lambda_X}\}.$$

We will call an ATI metric H on $\mathcal{E}$ a *reference metric* if it satisfies

$$\tilde{K}_H \in C^\infty_\delta(W, \mathrm{i}\,\mathfrak{su}(E, H)). \tag{4.2}$$

The existence of such metrics is guaranteed by the following lemma.

Lemma 10. *The bundle $(\mathcal{E},\theta)$ admits a reference metric.*

Proof. Since $(\mathcal{E}_X,\theta_X)$ is stable, it follows from Theorem 6 that $(\mathcal{E}_X,\theta_X)$ admits a HHE metric H_X. Denoting by $H_\times$ its pullback over the translation invariant bundle $(\mathcal{E}_\times,\theta_\times)$ (see Definition 16), define

$$H' = \phi \cdot H + (1-\phi)\cdot H_\times,$$

where H is a Hermitian metric over $\mathcal{E}$ and ϕ is a bump function satisfying

$$\phi|_{W_0} \equiv 1 \quad \text{and} \quad \phi|_{W\backslash W_1} \equiv 0.$$

By construction we have that $H' = H_\times$ over $W\backslash W_1$, hence it is ATI and satisfies

$$\tilde{K}_{H'} = \tilde{K}_{H_\times} = \pi^*\tilde{K}_{H_X} = 0 \quad \text{over} \quad W\backslash W_1.$$

Therefore H' is a reference metric on E. □

Now that we have a reference metric, we may adjust (4.1) to the case of metrics with exponential decay. For a reference metric H on $(\mathcal{E},\theta)$, consider the map

$$\mathfrak{L} : C^\infty_\delta(W, \mathrm{i}\,\mathfrak{su}(E,H)) \times [0,1] \to C^\infty_\delta(W, \mathrm{i}\,\mathfrak{su}(E,H))$$

given by

$$\mathfrak{L}(s,t) := \mathrm{Ad}(\mathrm{e}^{s/2})\tilde{K}_{H\,\mathrm{e}^s} + t\cdot s, \tag{4.3}$$

and set

$$\mathfrak{I} := \pi_2(\mathfrak{L}^{-1}(0)), \tag{4.4}$$

where $\pi_2 : C^\infty_\delta(W, \mathrm{i}\,\mathfrak{su}(E,H)) \times [0,1] \to [0,1]$ is the projection onto the second factor.

Remark 1. *As discussed in section 2.3, if we consider the map $\mathrm{e}^{s/2} : (E, H\,\mathrm{e}^s) \to (E,H)$ then the push-forward of the Hitchin-Simpson connection $D_{H\,\mathrm{e}^s}$ gives a connection $\tilde{D}_{H\,\mathrm{e}^s}$ such that*

$$i\Lambda_\omega F_{\tilde{D}_{H\,\mathrm{e}^s}} = \mathrm{Ad}(\mathrm{e}^{s/2})\tilde{K}_{H\,\mathrm{e}^s}.$$

Hence, since $\mathrm{e}^{s/2}$ is an isometry, this means that $\mathrm{Ad}(\mathrm{e}^{s/2})\tilde{K}_{H\,\mathrm{e}^s}$ is also $H-$self-adjoint. Moreover, since $\mathrm{e}^s \overset{\delta}{\rightsquigarrow} \mathrm{Id}$, it follows that $\mathrm{Ad}(\mathrm{e}^{s/2})\tilde{K}_{H\,\mathrm{e}^s}$ has asymptotic decay. Hence $\mathfrak{L}$ is indeed a well-defined map.

It is clear from the definition of $\mathfrak{I}$ that $0 \in \mathfrak{I}$ if and only if $\tilde{K}_{H\,\mathrm{e}^s} = 0$ for some $s \in C^\infty_\delta(W, \mathrm{i}\,\mathfrak{su}(E,H))$ which is precisely the condition for $H\,\mathrm{e}^s$ to be a weak HHE metric (Definition 12). Hence, if one shows that $0 \in \mathfrak{I}$, the existence of weak HHE metrics is established. With this in mind, the rest of the chapter is dedicated to prove is non-empty, open and closed, thus showing that $\mathfrak{I} = [0,1]$.

The following result is the first step in proving that $\mathfrak{I} = [0,1]$. For this we will show that after a particular change of reference metric one can obtain $1 \in \mathfrak{I}$. The idea of the proof follows a trick introduced by Itoh and Nakajima (ITOH; NAKAJIMA, 1990, page 440).

Proposition 13. *There exists a reference metric H_0 and a Hermitian endomorphism $s_1 \in C^\infty_\delta(W, \mathrm{i}\mathfrak{su}(E, H_0))$ such that $\mathfrak{L}(s_1, 1) = 0$.*

Proof. Let H be the reference metric asymptotic to the HHE metric H_X which was constructed in the proof of Lemma 10, and take

$$H_0 := H\,\mathrm{e}^{\tilde{K}_H}.$$

Since $H \overset{\delta}{\rightsquigarrow} H_X$, one can easily see that $H_0 \overset{\delta}{\rightsquigarrow} H_X$. Moreover, $\tilde{K}_H \in C^\infty_\delta(W, \mathrm{i}\,\mathfrak{su}(E, H))$ implies that $\tilde{K}_{H_0} \in C^\infty_\delta(W, \mathrm{i}\,\mathfrak{su}(E, H_0))$ (since $\mathrm{e}^{\tilde{K}_H} \overset{\delta}{\rightsquigarrow} \mathrm{Id}$). Thus H_0 is a reference metric.

Now, take

$$s_1 := -\tilde{K}_H.$$

Then $H_0 = H\,\mathrm{e}^{-s_1}$ and, since s_1 commutes with e^{-s_1}, we have

$$\begin{aligned}\langle s_1 \cdot u, v\rangle_{H_0} &= \langle \mathrm{e}^{-s_1} s_1 \cdot u, v\rangle_H = \langle s_1\,\mathrm{e}^{-s_1} \cdot u, v\rangle_H \\ &= \langle \mathrm{e}^{-s_1} \cdot u, s_1 \cdot v\rangle_H = \langle u, s_1 \cdot v\rangle_{H_0},\end{aligned}$$

for all $u, v \in \Gamma(E)$. Thus $s_1 \in C^\infty_\delta(W, \mathrm{i}\,\mathfrak{su}(\mathcal{E}, H_0))$ and satisfies

$$\begin{aligned}\mathfrak{L}(s_1, 1) &= \mathrm{Ad}(\mathrm{e}^{s_1/2})(\tilde{K}_{H_0\,\mathrm{e}^{s_1}}) + s_1 \\ &= \mathrm{Ad}(\mathrm{e}^{s_1/2})(\tilde{K}_H) + s_1 \\ &= -s_1 + s_1 = 0.\end{aligned}$$

Therefore H_0 and s_1 are the reference metric and endomorphism we were looking for. □

Based on this last result, from now on we will fix H_0 constructed above as the reference metric for $\mathfrak{L}$. Thus we have $1 \in \mathfrak{I}$.

4.2 $\mathfrak{I}$ is open

Having just proved that $\mathfrak{I}$ is non-empty, in this section we will show that $\mathfrak{I} \cap (0, 1]$ is open. For this we will use the implicit function theorem for Banach spaces to show that any solution (s, t) can be extended to a path of solutions $(t - \epsilon, t + \epsilon)$ for some $\epsilon > 0$.

We start observing that for $\mathfrak{L}$, as defined in (4.3), we can't make use of the implicit function theorem, since $C^\infty_\delta(W, \mathrm{i}\,\mathfrak{su}(E, H_0))$ is just a Frechet space (see Definition 18). Despite this, we have

$$\mathfrak{L}(s, t) = \mathrm{Ad}(\mathrm{e}^{s/2})(\tilde{K}_{H_0} + i\Lambda D''(\mathrm{e}^{-s}\,D'_{H_0}\,\mathrm{e}^{s})) + t \cdot s.$$

Thus, $\mathfrak{L}$ can be extended to a smooth map

$$\mathfrak{L} : C^{2,\alpha}_{\delta}(W, \mathrm{i}\,\mathfrak{su}(E, H_0)) \times [0,1] \to C^{0,\alpha}_{\delta}(W, \mathrm{i}\,\mathfrak{su}(E, H_0)),$$

where $C^{k,\alpha}_{\delta}(W, \mathrm{i}\,\mathfrak{su}(E, H_0))$ are Banach. This will be the appropriate setting to apply the implicit function theorem.

In the next lemma we show that the linearisation of $\mathfrak{L}$ is an isomorphism. For this we use the formulas in section 2.3 to compute the linearisation, and the asymptotic Fredholm properties of $\frac{1}{2}\Delta_{D_0} + t$ (see Proposition 12) to conclude that it is an isomorphism.

Lemma 11. *If $(s,t) \in C^{2,\alpha}_{\delta}(W, \mathrm{i}\,\mathfrak{su}(E, H_0)) \times (0,1]$ is a solution of $\mathfrak{L}(s,t) = 0$, then the linearisation*

$$L_{s,t} := \frac{\mathrm{d}\mathfrak{L}}{\mathrm{d}s}(s,t) : C^{2,\alpha}_{\delta}(W, \mathrm{i}\,\mathfrak{su}(E, H_0)) \to C^{0,\alpha}_{\delta}(W, \mathrm{i}\,\mathfrak{su}(E, H_0))$$

is an isomorphism.

Proof. We can write equation $\mathfrak{L}(s,t) = 0$ as

$$\mathrm{Ad}(\mathrm{e}^{s/2})\tilde{K}_{H_0\,\mathrm{e}^s} + t \cdot s = 0.$$

Thus, applying Proposition 7 and using that $\mathfrak{K}(s) = \mathrm{Ad}(\mathrm{e}^{s/2})\tilde{K}_{H_0\,\mathrm{e}^s} = -t \cdot s$, we obtain

$$L_{s,t}\hat{s} = \frac{1}{4}\tilde{D}_s^{\dagger}\tilde{D}_s(\mathrm{Id} + \mathrm{Ad}(\mathrm{e}^{-s/2}))\Upsilon(s/2)\hat{s} + \frac{t}{4}[s, (\mathrm{Id} - \mathrm{Ad}(\mathrm{e}^{-s/2})\Upsilon(s/2)\hat{s}] + t\hat{s},$$

where Υ as defined in (2.12) is given by

$$\Upsilon(s) := \frac{\mathrm{e}^{\mathrm{ad}_s} - \mathrm{Id}}{\mathrm{ad}_s}. \tag{4.5}$$

Now, note that $\gamma(x) := L_{x\cdot s,t}$ for $x \in [0,1]$ defines a path of linear operators such that

$$\gamma(0) = \frac{1}{2}D_0^{\dagger}D_0 + t \quad \text{and} \quad \gamma(1) = L_{s,t}.$$

Moreover, since $s \in C^{2,\alpha}_{\delta}(W, \mathrm{i}\,\mathfrak{su}(E, H_0))$, we have $\forall x \in [0,1]$ that $\gamma(x)$ is a bounded operator and

$$\gamma(x) \overset{\delta}{\rightsquigarrow} \frac{1}{2}(-\partial_l^2 - \partial_\theta^2 + D_{H_X}^{\dagger}D_{H_X} + 2t).$$

It follows from Proposition 11 that γ is a path of Fredholm operators. Hence, by the continuity of the index, we conclude that the index of $L_{s,t}$ is equal to that of $\frac{1}{2}D_0^{\dagger}D_0 + t$ which is zero by Theorem 9. Thus, to prove that $L_{s,t}$ is invertible we just have to show that its kernel is trivial.

First, note that

$$\int_W \langle L_{s,t}\hat{s}, (\mathrm{Id}+\mathrm{Ad}(\mathrm{e}^{-s/2}))\Upsilon(s/2)\hat{s}\rangle = \frac{1}{16}\int_W |\tilde{D}_s(\mathrm{Id}+\mathrm{Ad}(\mathrm{e}^{-s/2}))\Upsilon(s/2)\hat{s}|^2$$
$$+ t\int_W \langle \mathrm{ad}_{s/4}(\mathrm{Id}-\mathrm{Ad}(\mathrm{e}^{-s/2})\Upsilon(s/2)+\mathrm{Id})\hat{s}, (\mathrm{Id}+\mathrm{Ad}(\mathrm{e}^{-s/2}))\Upsilon(s/2)\hat{s}\rangle$$
$$\geqslant t\int_W \langle \mathrm{ad}_{s/4}(\mathrm{Id}-\mathrm{Ad}(\mathrm{e}^{-s/2})\Upsilon(s/2)+\mathrm{Id})\hat{s}, (\mathrm{Id}+\mathrm{Ad}(\mathrm{e}^{-s/2}))\Upsilon(s/2)\hat{s}\rangle$$

Thus, since $\left((\mathrm{Id}+\mathrm{Ad}(\mathrm{e}^{-s/2}))\Upsilon(s/2)\right)^* = \Upsilon(s/2)(\mathrm{Id}+\mathrm{Ad}(\mathrm{e}^{-s/2}))$ because s is H_0-self-adjoint, we have

$$\int_W \langle L_{s,t}\hat{s}, (\mathrm{Id}+\mathrm{Ad}(\mathrm{e}^{-s/2}))\Upsilon(s/2)\hat{s}\rangle \geqslant t\int_W \langle \Gamma(s)\hat{s},\hat{s}\rangle,$$

where

$$\Gamma(s) := \Upsilon(s/2)(\mathrm{Id}+\mathrm{Ad}(\mathrm{e}^{-s/2}))(\mathrm{ad}_{s/4}(\mathrm{Id}-\mathrm{Ad}(\mathrm{e}^{-s/2})\Upsilon(s/2)+\mathrm{Id}).$$

Now, for any $x\in\mathbb{R}$ we have

$$\frac{(\mathrm{e}^{x/2}-1)}{x/2}(1+\mathrm{e}^{-x/2})\left(\frac{x}{4}(1-\mathrm{e}^{-x/2})\frac{\mathrm{e}^{x/2}-1}{x/2}+1\right) = \frac{(\mathrm{e}^{x/2}-\mathrm{e}^{-x/2})(\mathrm{e}^{x/2}+\mathrm{e}^{-x/2})}{x}$$
$$= \frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{x} = \frac{2\sinh x}{x} \geqslant 2.$$

Hence, using (4.5), $\mathrm{Ad}(\mathrm{e}^s)=\mathrm{e}^{\mathrm{ad}_s}$ and $\mathrm{Spec}(\mathrm{ad}_s)\subset\mathbb{R}$, we conclude that

$$\int_W \langle L_{s,t}\hat{s}, \mathrm{Ad}(\mathrm{e}^{-s/2})\Upsilon(s/2)\hat{s}\rangle \geqslant 2t\int_W |\hat{s}|^2.$$

Therefore, if $\hat{s}\in\ker L_{s,t}$ we must have $|\hat{s}|=0$ which means that the kernel of $L_{s,t}$ is trivial. □

Now that we have all the conditions to apply the implicit function theorem, we need a regularity result guaranteeing that our solutions are smooth. This is done in the next lemma using a elliptic bootstrapping technique.

Lemma 12. *If $(s,t)\in C^{2,\alpha}_\delta(W,\mathrm{i}\,\mathfrak{su}(E,H_0))\times[0,1]$ is a solution of $\mathfrak{L}(s,t)=0$, then $s\in C^\infty_\delta(W,\mathrm{i}\,\mathfrak{su}(E,H_0))$.*

Proof. The equation $\mathfrak{L}(s,t)=0$ can be written as

$$\mathrm{Ad}(\mathrm{e}^{s/2})\tilde{K}_{H_0\,\mathrm{e}^s}+t\cdot s=0.$$

Thus, if we expand it using the formula in Proposition 6, we obtain

$$0 =(2-\cosh(\mathrm{ad}_{s/2}))\tilde{K}_{H_0}+\frac{1}{2}\Theta(s)\Delta_{D_0}s+t\cdot s$$
$$+\frac{i}{2}\Lambda(D_0\Upsilon(-s/2)\wedge D_0's)-\frac{i}{2}\Lambda(D_0\Upsilon(s/2)\wedge D_0''s)$$
$$-\frac{i}{4}\Lambda(\Upsilon(-s/2)D_0's\wedge\Upsilon(s/2)D_0''s+\Upsilon(s/2)D_0''s\wedge\Upsilon{-s/2}D_0's),$$

with Υ as defined in (4.5) and

$$\Theta(s) := \frac{\Upsilon(s/2) + \Upsilon(-s/2)}{2}. \tag{4.6}$$

Since

$$\frac{1}{2}\left(\frac{\mathrm{e}^{x/2} - 1}{x/2} + \frac{\mathrm{e}^{-x/2} - 1}{-x/2}\right) = \frac{1}{2}\frac{\mathrm{e}^{x/2} - \mathrm{e}^{-x/2}}{x/2} = \frac{\sinh(x/2)}{x/2} \geqslant 1,$$

we know that $\Theta(s)$ has eigenvalues greater than 1. Hence it is invertible, and we can rewrite the equation above as

$$\begin{aligned} 0 &= \Theta(s)^{-1}(2 - \cosh(\mathrm{ad}_{s/2}))\tilde{K}_{H_0} + \frac{1}{2}\Delta_{D_0} s + t \cdot \Theta(s)^{-1} s \\ &+ \Theta(s)^{-1}\frac{\mathrm{i}}{2}\Lambda(D_0\Upsilon(-s/2) \wedge D_0' s) - \Theta(s)^{-1}\frac{\mathrm{i}}{2}\Lambda(D_0\Upsilon(s/2) \wedge D_0'' s) \\ &- \Theta(s)^{-1}\frac{\mathrm{i}}{4}\Lambda(\Upsilon(-s/2)D_0' s \wedge \Upsilon(s/2)D_0'' s + \Upsilon(s/2)D_0'' s \wedge \Upsilon{-s/2}D_0' s). \end{aligned}$$

Therefore, we have

$$\frac{1}{2}\Delta_{D_0} s + A(s)(D_0 s \otimes D_0 s) + t \cdot \Theta(s)^{-1} s = \Theta(s)^{-1}(\cosh(\mathrm{ad}_{s/2}) - 2)\tilde{K}_{H_0}, \tag{4.7}$$

where A is a linear map depending on s.

Now, observe that (4.7) is a quasilinear elliptic equation and that the right hand side has the same regularity as the solution s itself. Therefore, we can use Theorem 15 inductively to conclude that $s \in C^\infty_\delta(W, \mathrm{i}\,\mathfrak{su}(E, H_0))$. □

Now that we have the two previous results, we are finally able to show that $\mathfrak{I} \cap (0,1]$ is open.

Proposition 14. *The set $\mathfrak{I} \cap (0,1]$ is open set.*

Proof. The idea is to show that every point in $\mathfrak{I} \cap (0,1]$ is an interior point. So, take $t \in \mathfrak{I} \cap (0,1]$ which is guaranteed to exist by Proposition 13. By the definition of $\mathfrak{I}$ there exists $s \in C^\infty_\delta(W, \mathrm{i}\mathfrak{su}(E, H_0))$ such that $\mathfrak{L}(s,t) = 0$. Thus, since $C^\infty_\delta(W, \mathrm{i}\mathfrak{su}(E, H_0)) \subset C^{2,\alpha}_\delta(W, \mathrm{i}\,\mathfrak{su}(E, H_0))$, it follows from Lemma 11 that (s,t) is a regular value of $\mathfrak{L}$ extended to $C^{2,\alpha}_\delta(W, \mathrm{i}\,\mathfrak{su}(E, H_0))$. By the implicit function theorem (Theorem 13) we know there is an $\epsilon > 0$ and a smooth map $h : (t - \epsilon, t + \epsilon) \to C^{2,\alpha}_\delta(W, \mathrm{i}\,\mathfrak{su}(E, H_0))$ such that

$$\mathfrak{L}(h(x), x) = \mathfrak{L}(s,t) = 0, \tag{4.8}$$

for all $x \in (t - \epsilon, t + \epsilon)$. Hence, it follows from Lemma 12 that $h(x) \in C^\infty_\delta(W, \mathrm{i}\mathfrak{su}(E, H_0))$ for all $x \in (t - \epsilon, t + \epsilon)$ which implies $(t - \epsilon, t + \epsilon) \subset \mathfrak{I}$. Therefore t is an interior point and we conclude that $\mathfrak{I} \cap (0,1]$ is open. □

4.3 $\mathfrak{I}$ is closed

In the previous sections we showed that, after choosing an adequate reference metric H_0, $\mathfrak{I} \cap (0,1]$ is a non-empty open subset of $[0,1]$. Following our strategy for the demonstration of Theorem 12, in this section we prove that $\mathfrak{I}$ is closed. The idea is to use an Arzelà-Ascoli type of result to a sequence of bounded solutions $\{(s_n, t_n)\}_{n\in\mathbb{N}}$ with $t_n \to t$, and obtain a converging subsequence whose limit (s,t) still satisfies $\mathfrak{L}(s,t) = 0$. To achieve this, we will prove that every solution is bounded in the $C^{k,\alpha}$ and $C^{k,\alpha}_\delta$ norms, with these bounds given by a priori estimates.

To keep the reading more comfortable, we start this section showing that $\mathfrak{I}$ is closed. Then, we relegate to the next subsections the proof of the $C^{k,\alpha}$ bounds (Proposition 18) and the $C^{k,\alpha}_\delta$ bounds (Proposition 21).

Proposition 15. *The set $\mathfrak{I}$ is closed.*

Proof. The idea is to show that every limit point of $\mathfrak{I}$ is in $\mathfrak{I}$. So, let t be a limit point of $\mathfrak{I}$ and $\{t_n\}_{n\in\mathbb{N}} \subset \mathfrak{I}$ a sequence converging to it. By the definition of $\mathfrak{I}$ we know there is a sequence $\{s_n\}_{n\in\mathbb{N}} \subset C^\infty_\delta(W, \mathrm{i}\mathfrak{su}(E,H_0))$ such that $\mathfrak{L}(s_n,t_n) = 0$ for all $n \in \mathbb{N}$. Using Proposition 21, we have constants $C_k > 0$ for all $k \in \mathbb{N}$ such that

$$\|s_n\|_{C^k_\delta} \leqslant C_k \tag{4.9}$$

for all $n \in \mathbb{N}$. Thus, it follows from Lemma 9 that there is a subsequence $\{s_{n_0}\}_{n_0\in\mathbb{N}} \subset C^0_\delta(W, \mathrm{i}\,\mathfrak{su}(E,H_0))$ converging in C^0 to a limit $\tilde{s}_0$.

Now, if we apply Lemma 9 to the subsequence $\{s_{n_0}\}_{n_0\in\mathbb{N}}$, we can find a subsequence $\{s_{n_1}\}_{n_1\in\mathbb{N}}$ of $\{s_{n_0}\}_{n_0\in\mathbb{N}}$ which converges in C^1 to a limit $\tilde{s}_1$. Repeating this process inductively, we are able to construct a chain of sequences $\{s_{n_0}\}_{n_0\in\mathbb{N}} \supset \cdots \supset \{s_{n_k}\}_{n_k\in\mathbb{N}} \supset \ldots$ each admitting a C^k limit $\tilde{s}_k$. In fact, we have $\tilde{s}_0 = \tilde{s}_k$ for all $k \in \mathbb{N}$ because the C^k norms are stronger than the C^0. Thus, taking the diagonal sequence $\{s_m\}_{m\in\mathbb{N}}$ whose mth term is the mth term of the mth subsequence, we have $s_m \xrightarrow{C^k} \tilde{s}_0$ for all $k \in \mathbb{N}$. Hence, $\tilde{s}_0 \in C^\infty_\delta(W, \mathrm{i}\mathfrak{su}(E,H_0))$ and, using the C^k convergence for $k \geqslant 2$, we obtain

$$\mathfrak{L}(\tilde{s}_0, t) = \mathfrak{L}(\lim_{m\to\infty} s_m, \lim_{m\to\infty} t_m) = \lim_{m\to\infty} \mathfrak{L}(s_m, t_m) = 0.$$

Therefore $(\tilde{s}_0, t)$ is a solution, which implies that $t \in \mathfrak{I}$. □

4.3.1 C^k bounds

In this subsection we establish the C^k bounds for solutions of $\mathfrak{L}(s,t) = 0$ which will be used later in the proof of the $C^{k,\alpha}_\delta$ bounds. Following the ideas of (JACOB; WALPUSKI, 2018), first we will show a C^0 bound depending only on (W, g, I) and H_0, and then, using the interior estimates of Theorem 17, we will obtain the C^k bounds. Let's start fixing some notation.

Throughout this section we will denote by (s,t), or simply s when the real factor is not relevant, a solution of $\mathfrak{L}(s,t)=0$. Since we will be mostly interested in the C^0 norm of s and how it behaves over the end of W, we write

$$\|s\|_p := \|s\|_{L^p(W)} \qquad \|s\|_{p,L} := \|s\|_{L^p(W\setminus W_L)} \qquad \|s\|_{p,\partial L} := \|s\|_{L^p(\partial W_L)}.$$

Lastly, denote by $L_0>0$ a large real number that will be fixed later.

Aiming to obtain an uniform C^0 bound for s, our first result asserts that the contribution of the end of W to the supremum of $|s|$ decays at least linearly.

Lemma 13. *There is a constant $C_1>0$ depending only on (W,g,I) and H_0 such that, for any $L>0$, the following inequality holds:*

$$\|s\|_\infty \leqslant \|s\|_{\infty,L} + C_1(L+1).$$

Proof. First notice that the above inequality follows trivially if $\|s\|_\infty = \|s\|_{\infty,L}$. Thus we will assume that $\|s\|_\infty > \|s\|_{\infty,L}$, which means that there is $x_0 \in W_L$ such that $|s|$ achieves its maximum, i.e., $|s|(x_0) = \|s\|_\infty$.

Substituting equation $\mathfrak{L}(s,t)=0$ in the formula of Proposition 8 we have

$$\langle -t\cdot s - \tilde{K}_{H_0}, s\rangle = \frac{1}{4}\Delta|s|^2 + \frac{1}{2}|\upsilon(-s)D_0 s|^2,$$

which gives

$$-4\langle \tilde{K}_{H_0}, s\rangle = \Delta|s|^2 + 2|\upsilon(-s)D_0 s|^2 + 4t|s|^2 \geqslant \Delta|s|^2.$$

Hence, we obtain

$$\Delta|s|^2 \leqslant 4\|s\|_\infty|\tilde{K}_{H_0}|. \tag{4.10}$$

Now, invoking Proposition 10, let $(f,A)\in C^{2,\alpha}_\delta(W)\oplus\mathbb{R}$ be the pair satisfying

$$\Delta(f - Al) = 4|\tilde{K}_{H_0}|,$$

where

$$\int_W \Delta f = 0 \quad\text{and}\quad A = \int_W \frac{4|\tilde{K}_{H_0}|}{\mathrm{vol}(S^1\times X)} > 0$$

By (4.10) we know that $|s|^2 - \|s\|_\infty(f-Al)$ is subharmonic, hence, applying the maximum principle (Lemma 17) on W_L, we have that

$$\begin{aligned}|s|^2(x_0) - \|s\|_\infty(f(x_0) - Al(x_0)) &\leqslant \sup_{x\in\partial W_L}(|s|^2 - \|s\|_\infty(f-Al))(x)\\ &\leqslant \|s\|^2_{\infty,L} + \|s\|_\infty(\sup_{x\in\partial W_L}(-f)(x) + AL).\end{aligned}$$

Rearranging the terms and using that $|s|(x_0) = \|s\|_\infty$ we get

$$\begin{aligned}\|s\|_\infty^2 - \|s\|_{\infty,L}^2 &\leqslant \|s\|_\infty(f(x_0) - Al(x_0) - \inf_{x\in\partial W_L} f(x) + AL) \\ &\leqslant \|s\|_\infty(AL + 2\|f\|_\infty).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\|s\|_\infty - \|s\|_{\infty,L} &\leqslant \frac{\|s\|_\infty}{\|s\|_\infty + \|s\|_{\infty,L}} A\left(L + \frac{2\|f\|_\infty}{A}\right) \\ &\leqslant \max(A, 2\|f\|_\infty)(L+1).\end{aligned}$$

□

Before we proceed to the next lemma, it is worth noting that if

$$\|s\|_{\infty,L} < 8\|f\|_{\infty,L} \quad \text{or} \quad \|s\|_\infty > 2\|s\|_{\infty,L}$$

for some $L > 0$, then it follows from Lemma 13 that

$$\|s\|_\infty \leqslant 8\|f\|_{\infty,L} + C_1(L+1) \quad \text{or} \quad \|s\|_\infty \leqslant 2C_1(L+1)$$

respectively. Thus, in both cases we have already the C^0 bound we are looking for. With this in mind, henceforth we will assume that $\|s\|_{\infty,L_0} \geqslant 8\|f\|_{\infty,L_0}$ and $\|s\|_\infty \leqslant 2\|s\|_{\infty,L_0}$.

Based on Lemma 13, our goal now is to obtain an uniform bound for $\|s\|_{\infty,L}$. The following result is the first step in this direction.

Lemma 14. *Let $L > 0$ and $x_0 \in \overline{W\backslash W_L}$ satisfy*

$$|s|(x_0) = \|s\|_{\infty,L},$$

then for all $L' \geqslant l(x_0)$ we have

$$\|s\|_{\infty,L} \leqslant 4\|s\|_{\infty,\partial L'} + 4A(L' - l(x_0)). \tag{4.11}$$

Proof. Recall from the proof of Lemma 13 that $|s|^2 - \|s\|_\infty(f - Al)$ is subharmonic. Thus, for any $L' \geqslant l(x_0)$, we can apply the maximum principle (Lemma 17) on $W_{L'}$ obtaining

$$\|s\|_{\infty,L}^2 - \|s\|_\infty f(x_0) + \|s\|_\infty Al(x_0) \leqslant \|s\|_{\infty,\partial L'}^2 + \|s\|_\infty\|f\|_{\infty,\partial L'} + \|s\|_\infty AL'.$$

Rearranging the terms and using the assumption that $\|s\|_{\infty,L} \geqslant 8\|f\|_{\infty,L}$ and $\|s\|_\infty \leqslant 2\|s\|_{\infty,L}$ we have

$$\begin{aligned}\|s\|_\infty A(l(x_0) - L') &\leqslant \|s\|_{\infty,\partial L'}^2 - \|s\|_{\infty,L}^2 + 2\|s\|_\infty\|f\|_{\infty,L} \\ &\leqslant \|s\|_\infty\left(\|s\|_{\infty,\partial L'} - \frac{\|s\|_{\infty,L}}{2} + \frac{\|s\|_{\infty,L}}{4}\right) \\ &= \|s\|_\infty\left(\|s\|_{\infty,\partial L'} - \frac{\|s\|_{\infty,L}}{4}\right),\end{aligned}$$

which gives

$$\|s\|_{\infty,L} \leqslant 4\|s\|_{\infty,\partial L'} + 4A(L' - l(x_0)).$$

$\square$

Take $L_0 > 0$ such that $(\mathcal{E},\theta)|_{X_z}$ is stable for $|z| \geqslant L_0$ and assume that $\|s\|_{\infty,L_0} \geqslant 32A$. Using the bound given in Lemma 14, we will now estimate $\|s\|_{\infty,L_0}$ in terms of $\tilde{K}_{H_0\,\mathrm{e}^s}$.

Proposition 16. *There is a constant $C_2 > 0$ depending only on (W,g,I) and H_0 such that the following inequality holds:*

$$\|s\|_{\infty,L_0}^{\frac{1}{2}} \leqslant C_2\|\tilde{K}_{H_0\,\mathrm{e}^s\,|X_z}\|_{2,L_0} + 1.$$

Proof. Set

$$r = \frac{\|s\|_{\infty,L_0}}{8A} \quad \text{and} \quad k = \lceil r \rceil,$$

where $\lceil x \rceil$ is denotes the lowest integer greater than $x \in \mathbb{R}$. If we take $l(x_0) \leqslant L \leqslant l(x_0) + k + 1$, it follows from Lemma 14 that

$$\begin{aligned}\|s\|_{\infty,L_0} &\leqslant 4\|s\|_{s,\partial L} + 4A(k+1) \leqslant 4\|s\|_{s,\partial L} + 4A(r+2)\\ &= 4\|s\|_{s,\partial L} + \frac{1}{2}\|s\|_{\infty,L_0} + 8A \leqslant 4\|s\|_{s,\partial L} + \frac{3}{4}\|s\|_{\infty,L_0},\end{aligned}$$

thus

$$\|s\|_{\infty,L_0} \leqslant 16\|s\|_{\infty,\partial L}.$$

As we have seen in the proof or Lemma 13, $\Delta|s|^2 \leqslant 4|\tilde{K}_{H_0}||s|$. Hence, using the inequality above and Theorem 16 it follows that

$$\begin{aligned}\|s\|_{\infty,L_0}^2 &\leqslant 16^2\|s\|_{\infty,\partial L}^2 \leqslant 16^2 \sup_{x \in W_{L+1}\backslash W_{L-1}} |s|^2\\ &\leqslant B\left(\int_{W_{L+1}\backslash W_{L-1}} |s|^2 + \int_{W_{L+1}\backslash W_{L-1}} 4|\tilde{K}_{H_0}||s|\right)\\ &\leqslant 4B\max(1,\mathrm{vol}(W_{L+1}\backslash W_{L-1}))\left(\int_{W_{L+1}\backslash W_{L-1}} |s|^2 + \mathrm{e}^{-\delta L_0}\|s\|_{\infty,L_0}\right),\end{aligned}$$

where $B > 0$ depends only on (W,g,I) and H_0. Setting

$$C = 4B \max_{L-l(x_0)=1,\dots,k}\{\max(1,\mathrm{vol}(W_{L+1}\backslash W_{L-1}))\}$$

and summing the inequality above over $L - l(x_0) = 1,\dots,k$ we have

$$\begin{aligned}\|s\|_{\infty,L_0}^3 = 8Ar\|s\|_{\infty,L_0}^2 &\leqslant 8AC\left(\int_{W_{l(x_0)+k+1}\backslash W_{l(x_0)}} |s|^2 + r\,\mathrm{e}^{-\delta L_0}\|s\|_{\infty,L_0}\right)\\ &\leqslant 8AC\left(\int_{W_{l(x_0)+k+1}\backslash W_{l(x_0)}} |s|^2\right) + C\,\mathrm{e}^{-\delta L_0}\|s\|_{\infty,L_0}^2.\end{aligned}$$

Hence, assuming $\|s\|_{\infty,L_0} \geqslant 2C$, we can rearrange the last term and obtain

$$\|s\|_{\infty,L_0}^{\frac{3}{2}} \leqslant 4\sqrt{AC}\|s\|_{L^2(W_{l(x_0)+k+1}\backslash W_{l(x_0)})}. \tag{4.12}$$

Now, since $(\mathcal{E},\theta)|_{X_z}$ is stable for $|z| > L_0$, we have a HHE metric H_{X_z} on each $(\mathcal{E},\theta)|_{X_z}$. Thus, if we set $\sigma_z := \log(H_{X_z}^{-1}H_0|_{X_z})$, it follows from Lemma 7 and Lemma 5 that

$$\begin{aligned}\|s|_{X_z}\|_{L^2(X_z)} &\leqslant C_z\|\log(\mathrm{e}^{\sigma_z}\,\mathrm{e}^{s|_{X_z}})\|_{L^2(X_z)} \\ &\leqslant C_z D_2 \mathcal{M}(H_{X_z}, H_{X_z}\,\mathrm{e}^{\sigma_z}\,\mathrm{e}^{s|_{X_z}}) + 1 \\ &\leqslant C_z D_2(\mathcal{M}(H_{X_z}, H_0|_{X_z}) + \mathcal{M}(H_0|_{X_z}, H_0\,\mathrm{e}^s\,|_{X_z}) + 1),\end{aligned}$$

where $C_z > 0$ depends on $\|\sigma_z\|_{L^2(X_z)}$. Notice that, by the way we constructed H_0, we have $\sigma_z \in C_\delta^\infty(W, i\mathfrak{su}(E,H_0))$. So that there is $C' > 0$ such that

$$\mathcal{M}(H_{X_z}, H_0|_{X_z}) \leqslant C'\,\mathrm{e}^{-\delta L_0}.$$

Hence, using the inequality above and Lemma 6 we get

$$\begin{aligned}\|s|_{X_z}\|_{L^2(X_z)} &\leqslant C_z D_2\left(C'\,\mathrm{e}^{-\delta L_0} + D_1\int_{X_z}|s||\tilde{K}_{H_0\,\mathrm{e}^s\,|_{X_z}}|\right) + 1 \\ &\leqslant C_z D_1 D_2 \|s\|_{\infty,L_0}\|\tilde{K}_{H_0\,\mathrm{e}^s\,|_{X_z}}\|_{L^2(X_z)} + C_z D_2 C' + 1.\end{aligned} \tag{4.13}$$

Integrating (4.13) on $W_{l(x_0)+k+1}\backslash W_{l(x_0)}$ we get

$$\begin{aligned}\|s\|_{L^2(W_{l(x_0)+k+1}\backslash W_{l(x_0)})} \leqslant\; &\tilde{C}\|s\|_{\infty,L_0}\|K_{H_0\,\mathrm{e}^s\,|_{X_z},\theta}\|_{L^2(W_{l(x_0)+k+1}\backslash W_{l(x_0)})} \\ &+ \tilde{C} + \mathrm{vol}(W_{l(x_0)+k+1}\backslash W_{l(x_0)}),\end{aligned}$$

where

$$\tilde{C} = D_2\int_{W_{l(x_0)+k+1}\backslash W_{l(x_0)}} C_z.$$

Therefore, using (4.12), we obtain

$$\|s\|_{\infty,L_0}^{\frac{1}{2}} \leqslant 4\sqrt{AC}\left(\tilde{C}\|K_{H_0\,\mathrm{e}^s\,|_{X_z},\theta}\|_{2,L_0} + \frac{\tilde{C}+1}{32A} + 1\right).$$

□

Now that we can estimate s through $\tilde{K}_{H_0\,\mathrm{e}^s}$, our next step is to show that $\tilde{K}_{H_0\,\mathrm{e}^s}$ is bounded.

Proposition 17. *There is a constant $C_3 > 0$ depending only on (W,g,I) and H_0 such that the following inequality holds:*

$$\|\tilde{K}_{H_0\,\mathrm{e}^s\,|_{X_z}}\|_{2,L_0}^2 \leqslant C_3\left(\mathrm{e}^{-\delta L_0} + \|\tilde{F}_{H_0}^\perp\|_{L^2(W_{L_0})}^2\right),$$

where $\tilde{F}_{H_0}^\perp$ is given by

$$\tilde{F}_{H_0}^\perp := \tilde{F}_{H_0} - i\Lambda\tilde{F}_{H_0}\omega.$$

Proof. Recall from Lemma 4 that for any Hermitian metric H on a Higgs bundle $(\mathcal{E},\theta)$ we have

$$\sigma_2(H)\wedge\omega^{n-2} = C\big(|\tilde{F}_H^{\perp}|^2 - |\tilde{K}_H|^2\big), \tag{4.14}$$

where

$$\sigma_2(H) := 2c_2(H) - \frac{r-1}{r}c_1(H)^2.$$

In the case of X_z which is compact, the integral of the left-hand side of (4.14) is a topological invariant of the bundle, thus we obtain

$$\int_{X_z}|\tilde{K}_{H_0\,\mathrm{e}^s\,|X_z}|^2 = \int_{X_z}|\tilde{K}_{H_0|X_z}|^2 + \int_{X_z}|\tilde{F}^{\perp}_{H_0\,\mathrm{e}^s\,|X_z}|^2 + \int_{X_z}|\tilde{F}^{\perp}_{H_0|X_z}|^2.$$

Now, by the way we constructed H_0 (remember that H_0 is translation-invariant for $|z|\gg 1$) we have

$$|\tilde{K}_{H_0|X_z}| \leqslant C'\,\mathrm{e}^{-\delta|z|} \quad \text{and} \quad |\tilde{F}^{\perp}_{H_0}| = |\tilde{F}^{\perp}_{H_0|X_z}|,$$

for $C'>0$. Moreover, since $s\in C^{\infty}_{\delta}(W, i\mathfrak{su}(E,H_0))$ we also get

$$|\tilde{F}^{\perp}_{H_0\,\mathrm{e}^s} - \tilde{F}^{\perp}_{H_0\,\mathrm{e}^s\,|X_z}| \leqslant \tilde{C}\,\mathrm{e}^{-\delta|z|},$$

for $\tilde{C}>0$. It follows that

$$\begin{aligned}\int_{X_z}|\tilde{K}_{H_0\,\mathrm{e}^s\,|X_z}|^2 &\leqslant \int_{X_z}|\tilde{F}^{\perp}_{H_0\,\mathrm{e}^s\,|X_z}|^2 + \int_{X_z}|\tilde{F}^{\perp}_{H_0|X_z}|^2 + C'\,\mathrm{e}^{-\delta|z|}\\ &\leqslant \int_{X_z}|\tilde{F}^{\perp}_{H_0\,\mathrm{e}^s}|^2 + \int_{X_z}|\tilde{F}^{\perp}_{H_0}|^2 + (C'+\tilde{C})\,\mathrm{e}^{-\delta|z|},\end{aligned}$$

which integrating over $W\backslash W_{L_0}$ produces

$$\begin{aligned}\|\tilde{K}_{H_0\,\mathrm{e}^s\,|X_z}\|^2_{2,L_0} &\leqslant \int_{W\backslash W_{L_0}}|\tilde{F}^{\perp}_{H_0\,\mathrm{e}^s\,|X_z}|^2 - |\tilde{F}^{\perp}_{H_0|X_z}|^2 + (C'+\tilde{C})\,\mathrm{e}^{-\delta L_0}\\ &\leqslant \int_{W}|\tilde{F}^{\perp}_{H_0\,\mathrm{e}^s\,|X_z}|^2 - |\tilde{F}^{\perp}_{H_0|X_z}|^2 + (C'+\tilde{C})\,\mathrm{e}^{-\delta L_0} + \|\tilde{F}^{\perp}_{H_0|X_z}\|^2_{L^2(W_{L_0})}.\end{aligned} \tag{4.15}$$

Now, since $s\in C^{\infty}_{\delta}(W, i\mathfrak{su}(E,H_0))$, we have

$$\int_W(\sigma_2(H_0\,\mathrm{e}^s) - \sigma_2(H_0))\wedge\omega^{n-2} = 0.$$

Thus, we can apply (4.14) and get

$$\int_W|\tilde{F}^{\perp}_{H_0\,\mathrm{e}^s}|^2 - |\tilde{F}^{\perp}_{H_0}|^2 = \int_W|\tilde{K}_{H_0\,\mathrm{e}^s}|^2 - |\tilde{K}_{H_0}|^2.$$

Using $\mathfrak{L}(s,t)=0$ and (4.11), we obtain

$$\int_W|\tilde{K}_{H_0\,\mathrm{e}^s}|^2 = \int_W t^2|s|^2 \leqslant \int_W t|\tilde{K}_{H_0}||s| - \frac{t}{4}\Delta|s|^2.$$

Denoting by N_L the outward oriented unit vector field normal to ∂W_L, we have

$$\int_W \frac{t}{4}\Delta|s|^2 = \frac{t}{4}\lim_{L\to\infty}\int_{W_L}\Delta|s|^2 = \frac{t}{4}\lim_{L\to\infty}\int_{\partial W_L} N_L|s|^2.$$

Thus, since in the tubular model N_L can be identified with ∂_l and $s \in C^\infty_\delta(W, \mathfrak{isu}(E, H_0))$, it follows that

$$|N_L|s|^2| \leqslant 2|N_L||\nabla_0 s||s| \leqslant \|N_L\|_\infty \|s\|^2_{C^1_{\delta,*}} \mathrm{e}^{-2\delta L},$$

which implies that

$$\frac{t}{4}\lim_{L\to\infty}\int_{\partial W_L} N_L|s|^2 = 0.$$

Therefore we have

$$\begin{aligned}\int_W |\tilde{K}_{H_0\,\mathrm{e}^s}|^2 &\leqslant \int_W t|\tilde{K}_{H_0}||s| \leqslant \int_W \frac{1}{2}|\tilde{K}_{H_0}|^2 + \frac{1}{2}t^2|s|^2\\ &\leqslant \int_W \frac{1}{2}|\tilde{K}_{H_0}|^2 + \frac{1}{2}|\tilde{K}_{H_0\,\mathrm{e}^s}|^2,\end{aligned}$$

which implies

$$\int_W |\tilde{F}^\perp_{H_0\,\mathrm{e}^s}|^2 - |\tilde{F}^\perp_{H_0}|^2 \leqslant 0. \tag{4.16}$$

Hence, comparing (4.15) and (4.16) completes the proof. □

Now that Proposition 17 is proved, we are finally able to show the C^0 bound for the solutions of $\mathfrak{L}(s,t) = 0$. Moreover, using interior estimates we can extend this to C^k bounds.

Proposition 18. *For all $k \in \mathbb{N}$ there is $c_k \in \mathbb{R}$ such that*

$$\|s\|_{C^k} \leqslant C_k,$$

for any $(s,t) \in C^\infty_\delta(W, \mathfrak{isu}(E,H_0)) \times [0,1]$ solution of $\mathfrak{L}(s,t) = 0$.

Proof. The assumptions we made about s along this section were

1. $\|s\|_{\infty,L} < 8\|f\|_{\infty,L_0}$ or $\|s\|_\infty > 2\|s\|_{\infty,L_0}$;
2. $\|s\|_{\infty,L_0} \geqslant 32A$;
3. $\|s\|_{\infty,L_0} \geqslant 2C$.

Hence, for s not fitting these cases we can use Lemma 13 to obtain

$$\|s\|_\infty \leqslant \max\{8\|f\|_{\infty,L_0}, 32A, 2C\} + 2C_1(L_0+1).$$

On the other hand, for s satisfying all these conditions it follows from Lemma 13, Proposition 16 and Proposition 17 that

$$|s|_\infty \leqslant \left(C_2\sqrt{C_3(\mathrm{e}^{-\delta L_0} + \|\tilde{F}^\perp_{H_0}\|^2_{L^2(W_{L_0})})} + 1\right) + C_1(L_0+1).$$

Thus, for any solution s we have an upper bound for $\|s\|_{C^0}$ given by the maximum of the above inequalities. Now that we have C^0 bounds, the C^k bounds follow from Theorem 17. □

4.3.2 $C^{k,\alpha}_\delta$ bounds

In this subsection we show the $C^{k,\alpha}_\delta$ bounds for solutions of $\mathfrak{L}(s,t) = 0$ that are used in the proof of Proposition 15. The proof consists in showing locally an uniform $C^{k,\alpha}_\epsilon$ bound using Schauder estimates, for some well chosen $\epsilon > 0$, and then using an inductive process to improve the ϵ decay to a δ decay.

Before we proceed to prove the $C^{k,\alpha}_\delta$ bounds, we will show a simple analytical lemma that will be useful later.

Lemma 15. *Suppose that a continuous function $f : [0,\infty) \to [0,\infty)$ is C^1 over $(0,\infty)$ and satisfies*

$$f(x) \leqslant A\,\mathrm{e}^{-\delta x} - Bf'(x)$$

for some positive constants $A, B \in \mathbb{R}$. Then, taking $\epsilon := \min\{\delta, 1/2B\}$, we have

$$f(x) \leqslant (2A + f(0))\,\mathrm{e}^{-\epsilon x}.$$

Proof. If one defines the function $g : [0,\infty) \to \mathbb{R}$ by

$$g(x) := f(x) - (2A + f(0))\,\mathrm{e}^{-\epsilon x},$$

it follows that $g(0) = -2A \leqslant 0$ and

$$\begin{aligned} Bg'(x) &= Bf'(x) + B\epsilon(2A + f(0))\,\mathrm{e}^{-\epsilon x} \\ &\leqslant A\,\mathrm{e}^{-\delta x} - f(x) + \frac{1}{2}(2A + f'(0))\,\mathrm{e}^{-\epsilon x} \\ &\leqslant -f(x) + \frac{1}{2}(2A + f'(0))\,\mathrm{e}^{-\epsilon x} = -g(x). \end{aligned}$$

Thus, taking

$$h(x) = g'(x) + \frac{g(x)}{B},$$

it is easy to see that

$$g(x) = \mathrm{e}^{-\frac{x}{B}}\left(-2A + \int_0^x \mathrm{e}^{\frac{s}{B}}\, h(s)ds\right) \leqslant 0$$

since $h(x) \leqslant 0$. □

To use the Schauder estimates and obtain a $C^{k,\alpha}_\epsilon$–bound on s, we need an uniform L^2_ϵ–bound for every solution of $\mathfrak{L}(s,t) = 0$. The following result, along with the next remark, provides us with these L^2_ϵ–bounds at least locally. Like in the previous section, fix $L_0 \gg 1$ such that $(\mathcal{E},\theta)|_{X_z}$ is stable for $|z| \geqslant L_0$.

Proposition 19. *There are constants $\epsilon > 0$ and $C_\epsilon > 0$, depending on (W, g, I) and $(\mathcal{E}, \theta)$, such that, for all $L \geqslant L_0$, we have*

$$\int_{W\backslash W_L} |s|^2 \leqslant C_\epsilon \,\mathrm{e}^{-2\epsilon L} \quad \textit{and} \quad \int_{W\backslash W_L} |D_0 s|^2 \leqslant C_\epsilon \,\mathrm{e}^{-2\epsilon L}.$$

Proof. It follows from Lemma 3 and our choice of L_0 that

$$\int_{\partial W_L} |s|^2 \leqslant C_1 \int_{\partial W_L} |D_0 s|^2 \quad \forall L \geqslant L_0, \tag{4.17}$$

where C_1 just depends on (W, g, I) and $(\mathcal{E}, \theta)$. Thus, to prove this result we just need to show the second estimate.

Applying Proposition 8 in the equation $\mathfrak{L}(s,t) = 0$ we see that

$$\Delta|s|^2 + 2|\upsilon(-s) D_0 s|^2 \leqslant -4\langle \tilde{K}_{H_0}, s\rangle,$$

where

$$\upsilon(-s) = \sqrt{\Upsilon(-s)} = \sqrt{\frac{\mathrm{Id} - \mathrm{e}^{-\operatorname{ad}_s}}{\operatorname{ad}_s}}.$$

Since

$$\lim_{x\to 0} \frac{1-\mathrm{e}^{-x}}{x}(1+x) = 1,$$

we know there is $a > 0$ such that the expression above is greater than $\frac{1}{4}$ for all $x \in [0, a]$. Moreover, for all $x \geqslant a$ we have

$$\frac{1-\mathrm{e}^{-x}}{x}(1+x) \geqslant 1 - \mathrm{e}^{-a}.$$

Hence, it follows that

$$\sqrt{\frac{1-\mathrm{e}^{-x}}{x}} \geqslant \frac{M}{\sqrt{1+|x|}},$$

for $M = \min(\frac{1}{2}, 1-\mathrm{e}^{-a})$. Using this estimate for $\upsilon(-s)$ and the fact that $\|s\|_{L^\infty} < \infty$, we conclude that

$$|D_0 s|^2 \leqslant \frac{M}{2}(1 + \|s\|_{L^\infty})(4|\tilde{K}_{H_0}||s| - \Delta|s|^2). \tag{4.18}$$

Now, integrating (4.18) over $W_{L'}\backslash W_L$, for $L' > L$, and using that $\tilde{K}_{H_0} \in C^\infty_\delta(W, i\mathfrak{su}(E, H_0))$ along with Proposition 18 yields

$$\int_{W_{L'}\backslash W_L} |D_0 s|^2 \leqslant \frac{M}{2}(1+\|s\|_{L^\infty})\left(4\|\tilde{K}_{H_0}\|_{C^0_\delta}\|s\|_{C^0}\,\mathrm{e}^{\delta L} - \int_{\partial W_{L'}} N_{L'}|s|^2 + \int_{\partial W_L} N_L|s|^2\right),$$

where $N_L, N_{L'}$ are the outward oriented unit vector field normal to $\partial W_L, \partial W_{L'}$. Just as in the proof of Proposition 17, we have

$$|N_{L'}|s|^2| \leqslant 2|N_{L'}||\nabla_0 s||s| \leqslant \|N_{L'}\|_\infty \|s\|^2_{C^1_\delta}\,\mathrm{e}^{-2\delta L'},$$

since $s \in C^\infty_\delta(W, \mathfrak{isu}(E, H_0))$. Thus taking the limit $L' \to \infty$ and using (4.17) gives

$$\begin{aligned}\int_{W\backslash W_L} |D_0 s|^2 &\leqslant C_2\left(\mathrm{e}^{-\delta L} + \int_{\partial W_L} |\nabla_0 s||s|\right)\\ &\leqslant C_2\left(\mathrm{e}^{-\delta L} + \int_{\partial W_L} (|D_0 s| + |(\theta+\theta^*)s|)|s|\right)\\ &\leqslant C_2\left(\mathrm{e}^{-\delta L} + (C_1^{\frac{1}{2}} + C_1\|\theta+\theta^*\|_{C^0})\int_{\partial W_L} |D_0 s|^2\right),\end{aligned}$$

where $C_2 = \dfrac{M}{2}(1 + \|s\|_{L^\infty})\max(4\|\tilde{K}_{H_0}\|_{C^0_\delta}\|s\|_{C^0}, 2\|N_L\|_\infty)$. The result now follows from Lemma 15. □

Remark 2. *Note that, since (W, g, I) is ACyl, there is $L_0' \geqslant L_0$ such that for all $x \in W\backslash W_{L_0'}$ and all $r < \mathrm{inj}(W)$ we have $B_r(x) \subset W_{l(x)+2r}\backslash W_{l(x)-2r}$. Thus, it follows from Proposition 19 that for all $x \in W\backslash W_{L_0'}$ we have*

$$\begin{aligned}\int_{B_r(x)} |\mathrm{e}^{\epsilon l}\, s|^2 \leqslant \int_{W_{l(x)+2r}\backslash W_{l(x)-2r}} |\mathrm{e}^{\epsilon l}\, s|^2 \leqslant \mathrm{e}^{2\epsilon(l(x)+2r)} \int_{W\backslash W_{l(x)-2r}} |s|^2\\ \leqslant C_\epsilon\, \mathrm{e}^{\epsilon 8r} < C_\epsilon\, \mathrm{e}^{8\epsilon\, \mathrm{inj}(W)}.\end{aligned}$$

Moreover, by Proposition 18, we have

$$\int_{B_r(x)} |\mathrm{e}^{\epsilon l}\, s|^2 \leqslant \mathrm{e}^{2\epsilon(L_0'+\mathrm{inj}(W))}\, C_0^2\, \mathrm{vol}(W_{L_0'+\mathrm{inj}(W)}),$$

for every $x \in W_{L_0'}$. Therefore there is a constant $\tilde{C}_\epsilon$ depending on (W, g, I), ϵ and C_ϵ such that for any $x \in W$ and $r < \mathrm{inj}(W)$ we have

$$\int_{B_r(x)} |\mathrm{e}^{\epsilon l}\, s|^2 \leqslant \tilde{C}_\epsilon.$$

Now that we have the L^2 bounds for $\mathrm{e}^{\epsilon l}\, s$ from Remark 2, we can use the estimates of Proposition 18 and obtain $C^{k,\alpha}_\epsilon$ bounds for s.

Proposition 20. *For all $k \in \mathbb{N}$ and $0 < \alpha \leqslant 1$ there is $C_{k,\alpha,\epsilon} \in \mathbb{R}$ such that*

$$\|s\|_{C^{k,\alpha}_\epsilon} \leqslant C_{k,\alpha,\epsilon},$$

for any $(s, t) \in C^\infty_\delta(W, \mathfrak{isu}(E, H_0)) \times [0, 1]$ solution of $\mathfrak{L}(s, t) = 0$.

Proof. Using Proposition 6, we can write equation $\mathfrak{L}(s, t) = 0$ as

$$\Delta_{D_0} s + A(s)(D_0 s \otimes D_0 s) + ts = \Theta(s)^{-1}(\cosh(\mathrm{ad}_{s/2}) - 2)\tilde{K}_{H_0},$$

where A is linear map depending on s and

$$\Theta(s) := \frac{\Upsilon(s/2) + \Upsilon(-s/2)}{2}.$$

Let $P_s : \Gamma(\operatorname{End} E) \to \Gamma(\operatorname{End} E)$ be the elliptic linear differential operator given by

$$P_s(u) = \Delta_{D_0} u + A(s)(D_0 s \otimes D_0 u) + t \cdot u$$

and take $P_s' = \mathrm{e}^{\epsilon l} \circ P_s \circ \mathrm{e}^{-\epsilon l}$. Doing some computations one sees that

$$\begin{aligned} P_s'(u) &= \Delta_{D_0} u + A(s)(D_0 s \otimes D_0 u) + \epsilon\langle dl, D_0 u\rangle \\ &\quad + A(s)(D_0 s \otimes u) + \frac{1}{2}(\epsilon^2 |dl| - \epsilon \Delta l)u + tu \end{aligned}$$

so that, using Proposition 18, we have

$$\|P_s'\|_{k,\alpha} \leqslant c_{k,\alpha,\epsilon}$$

with $c_{k,\alpha,\epsilon}$ independent of the solution s.

Now, take $x \in W$ and consider, for $r < \operatorname{inj}(W)$, a normal coordinate system $\phi_x : B_r(x) \to B_r(0)$ along with a synchronous trivialization (Definition 22). Using Corollary 5 and the Sobolev embedding we have

$$\|\mathrm{e}^{\epsilon l}\, s\|_{C^{k\alpha}(B_r(x))} \leqslant a_{k,\alpha}^{-1} \|\phi_x^* \,\mathrm{e}^{\epsilon l}\, s\|_{C^{k\alpha}(B_r(0))} \leqslant c_{k,\alpha,r}' \|\phi_x^* \,\mathrm{e}^{\epsilon l}\, s\|_{W^{k+n,2}(B_r(0))},$$

and it follows from elliptic interior estimates Theorem 14 that

$$\|\phi_x^* \,\mathrm{e}^{\epsilon l}\, s\|_{W^{k+n,2}(B_r(0))} \leqslant C\big(\|\phi_x^*(P_s' \,\mathrm{e}^{\epsilon l}\, s)\|_{W^{k+n-2,2}(B_r(0))} + \|\phi_x^* \,\mathrm{e}^{\epsilon l}\, s\|_{L^2(B_r(0))}\big)$$

with C depending on $\|P_s'\|_{k+n-1}$, k, n and r. Since $\tilde{K}_{H_0} \in C_\delta^\infty(W, \mathfrak{i}\mathfrak{su}(E, H_0))$, we have

$$\begin{aligned} \|P_s'(\mathrm{e}^{\epsilon l}\, s)\|_{C^{k,\alpha}} &= \|\mathrm{e}^{\epsilon l}\, \Theta(s)^{-1}(\cosh(\mathrm{ad}_{s/2}) - 2)\tilde{K}_{H_0}\|_{C^{k,\alpha}} \\ &\leqslant \|\Theta(s)^{-1}(\cosh(\mathrm{ad}_{s/2}) - 2)\|_{C^{k,\alpha}} \|\tilde{K}_{H_0}\|_{C_\delta^{k,\alpha}}, \end{aligned}$$

and, by Proposition 18, the term depending on s is uniformly bounded by say $A_{k,\alpha}$. Thus, it follows from Remark 2 that

$$\|\mathrm{e}^{\epsilon l}\, s\|_{C^{k,\alpha}(B_r(x))} \leqslant C c_{k,\alpha,r}' \left(A_{k,\alpha} \|\tilde{K}_{H_0}\|_{C_\delta^{k,\alpha}} + \tilde{C}_\epsilon\right),$$

for all $x \in W$ which implies that $\|s\|_{C_\epsilon^{k,\alpha}}$ is bounded independent of s. □

Finally, using Proposition 20 and Remark 2 we are able to show the $C_\delta^{k,\alpha}-$bounds for solutions of $\mathfrak{L}(s,t) = 0$.

Proposition 21. *For all $k \in \mathbb{N}$ and $0 < \alpha \leqslant 1$ there is $C_{k,\alpha} \in \mathbb{R}$ such that*

$$\|s\|_{C_\delta^{k,\alpha}} \leqslant C_{k,\alpha},$$

for any $(s,t) \in C_\delta^\infty(W, \mathfrak{i}\mathfrak{su}(E, H_0)) \times [0,1]$ solution of $\mathfrak{L}(s,t) = 0$.

Proof. First, observe that

$$\|D_0 s \otimes D_0 s\|_{C^{k,\alpha}_{2\epsilon}} \leqslant \mathrm{e}^k \|D_0 s\|^2_{C^{k,\alpha}_{\epsilon}},$$

which is uniformly bounded by Proposition 20. So, using that $\Theta(s)^{-1}(\cosh(\mathrm{ad}_{s/2}) - 2)\tilde{K}_{H_0} \in C^\infty_\delta(W, \mathrm{i}\mathfrak{su}(E, H_0))$ and $\epsilon < \delta$, we obtain

$$\|\Delta_{D_0} s + ts\|_{C^{k,\alpha}_{\epsilon'}} \leqslant \mathrm{e}^k\, C^2_{k,\alpha,\epsilon} + A_{k,\alpha}\|\tilde{K}_{H_0}\|_{C^{k,\alpha}_{\delta}} = B_{k,\alpha,\epsilon'},$$

with $\epsilon' := \min\{2\epsilon, \delta\}$. In particular, we have

$$|\Delta_{D_0} s + ts| \leqslant B_{0,1,\epsilon'}\, \mathrm{e}^{\epsilon' l}$$

which implies that

$$|D_0 s|^2 \leqslant |\Delta_{D_0} s||s| \leqslant B_{0,1,\epsilon'} C_{0,1,\epsilon}\, \mathrm{e}^{(\epsilon+\epsilon')l}\,.$$

Thus we have

$$\int_{W\setminus W_L} |D_0 s|^2 \leqslant C\, \mathrm{e}^{-(\epsilon+\epsilon')L},$$

and following the same deduction in Remark 2 and the proof of Proposition 20 above, we may conclude that $\|s\|_{C^{k,\alpha}_{(\epsilon+\epsilon')/2}}$ is bounded independent of s. Applying this same process inductively we arrive at a bound having δ decay concluding our result. $\square$

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APPENDIX A – Some technical analytical resuts

In this appendix I list some technical analytical results that are used along the text but may have no direct interest to the reader.

Lemma 16 (General Leibniz rule)**.** *Let E_1, E_2 be vector bundles over X and ∇_1, ∇_2 connections on E_1, E_2 respectively. If $s_1 \in \Gamma(E_1)$ and $s_2 \in \Gamma(E_2)$ then*

$$\nabla^n(s_1 \otimes s_2) = \sum_{k=0}^{n} \binom{n}{k} \nabla^{n-k} s_1 \otimes \nabla^k s_2,$$

where $\nabla = \nabla_1 \otimes \nabla_2$.

Lemma 17 (Maximum principle)**.** *Let (X, g) be a compact Riemannian manifold with boundary and suppose that $f \in C^2(X)$ is a nonnegative function satisfying*

$$\Delta f(x) \leqslant 0$$

for all $x \in X$. Then we have

$$\sup_{X \setminus \partial X} f(x) \leqslant \sup_{\partial X} f(x)$$

with equality only if f is constant.

Theorem 13 ((LANG, 1993, Theorem 2.1, page 364))**.** *Let E_1, E_2 and F be Banach spaces, $U \subset E_1 \times E_2$ be an open subset, and $f : U \to F$ a smooth map. If the partial derivative $(D_1 f)$ at a point $(\xi_1, \xi_2) \in U$ is an isomorphism from E_1 to F there is a smooth map h from a neighbourhood of $\xi_2 \in E_2$ to a neighbourhood of $\xi_1 \in E_1$ such that*

$$f(h(\eta), \eta) = f(\xi_1, \xi_2).$$

Let $E = \underline{\mathbb{C}}^r$ be the trivial bundle over $\mathbb{R}^n$ and denote by $\langle \cdot, \cdot \rangle$ and ∂ the trivial metric and connection of E. For an elliptic operator of order m

$$L = \sum_{|\alpha| \leqslant m} A_\alpha(x) \partial^\alpha : C^\infty(E) \to C^\infty(E),$$

$k \in \mathbb{N}$ and $R > 0$ we define

$$\|L\|_{k,R} = \sum_{|\alpha| \leqslant m} \|A_\alpha\|_{C^{k+m-|\alpha|}(B_R(0))}.$$

Theorem 14 ((NICOLAESCU, 2007, Theorem 10.3.1, page 465)). *Let $(k,p) \in \mathbb{N} \times (1,\infty)$, and $R > 0$. Then, there exists $C = C(\|L\|_{k+1}, k, p, n, R) > 0$ such that, $\forall u \in C_0^\infty(E|_{B_R(0)})$, we have*

$$\|u\|_{k+m,p} \leqslant C(\|Lu\|_{k,p} + \|u\|_p). \tag{A.1}$$

Moreover, if $\alpha \in (0,1)$ then there exists $C' = C'(\|L\|_{k+1}, k, \alpha, n, R) > 0$ such that

$$\|u\|_{k+m,\alpha} \leqslant C(\|Lu\|_{k,\alpha} + \|u\|_{0,\alpha}). \tag{A.2}$$

Theorem 15 ((AUBIN, 1998, Theorem 3.56, page 86)). *Let $A(u) = F(x, u, \nabla u, \nabla^2 u)$ be a differential operator of order two, defined on Ω an open set of $\mathbb{R}^n$, F being a smooth function of its arguments. Suppose that A is elliptic on Ω at $u_0 \in C^2(\Omega)$, and that $A(u_0) = f \in C^{r,\beta}(\Omega)$ with $0 < \beta < 1$. Then $u_0 \in C^{r+2,\beta}(\Omega)$.*

Theorem 16 ((GILBARG; TRUDINGER, 2001, Theorem 9.20, page 244)). *Let $W^{2,n}(\Omega)$ and suppose that $Lu \geqslant f$, where $f \in L^n(\Omega)$. Then for any ball $B = B_{2R}(y) \subset \Omega$ and $p > 0$, we have*

$$\sup_{B_R(y)} \leqslant C\left\{\left(\frac{1}{|B|}\int_B (u^+)^p\right)^{\frac{1}{p}} + \frac{R}{\lambda}\|f\|_{L^n(B)}\right\}$$

where $C = C(n, \gamma, \nu R^2, p)$.

Theorem 17 ((JACOB; WALPUSKI, 2018, Theorem C.1, page 1589)). *Let (X, g, I) be a compact Kähler manifold of dimension n with bounded geometry and let $\mathcal{E}$ be a holomorphic vector bundle over X. If H_0 and H are Hermitian metrics on $\mathcal{E}$ and $s := \log(H_0^{-1}H) \in C^\infty(X, i\mathfrak{su}(E, H_0))$, then*

$$\begin{aligned} r^{k+2-\frac{2n}{p}}\|\nabla_{H_0}^{k+2}s\|_{L^p(B_r(x))} &\leqslant \epsilon_{k,p}\big(\|s\|_{L^\infty(B_{2r}(x))} + \|K_H\|_{L^\infty(B_{2r}(x))}\big) \\ &\quad + r^{k-\frac{2n}{p}}\|\nabla^k K_H\|_{L^p(B_{2r}(x))} + \sum_{j=0}^{k} r^{2+i}\|\nabla_{H_0}^i F_{H_0}\|_{L^\infty(B_{2r}(x))}, \end{aligned}$$

where $\epsilon_{k,p}$ is a smooth function which vanishes at the origin and depends only on $k \in \mathbb{N}$, $p \in (1,\infty)$, and the geometry of X.

For proof of the following theorem see (LANG, 1993, Theorem 3.1, page 57)

Theorem 18 (Arzelà-Ascoli). *Let X be a compact subset of a metric space, and F be a Banach space. Let Φ be a subset of the space of continuous maps $C(X,F)$ with sup norm. Then Φ is relatively compact in $C(X,F)$ if and only if the following two conditions are satisfied:*

- *For every $x \in X$ and $\epsilon > 0$ $\exists \delta = \delta(x,\epsilon) > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $y \in B_\delta(x)$.*
- *For each $x \in X$, the set $\{f(x) : f \in \Phi\}$ is relatively compact.*

Corollary 6. *Suppose that* $\Phi \subset C^{k+1}(B_r(0), \mathbb{R})$ *satisfies*

$$\|f\|_{C^{k+1}} \leqslant C \quad \forall f \in \Phi$$

for some $C \in \mathbb{R}$*. Then* Φ *is relatively compact as a subset of* $C^k(B_r(0), \mathbb{R})$*.*